Conformal Mapping of a Rotational Ellipsoid to a Sphere

On the occasion of the proclamation of C. F. Gauss as the global surveyor in 2021

Konforme Abbildung eines Rotationsellipsoids auf eine Kugel

Anlässlich der Proklamation von C. F. Gauß zum globalen Geodäten 2021

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Summary

Carl Friedrich Gauss is especially known in geodesy for the Gauss-Krüger map projection, which is in official use in many countries and is sometimes known as the transverse Mercator projection. Gauss also investigated other variants of conformal mappings of a rotational ellipsoid to a sphere and of a sphere and an ellipsoid into a plane. In this paper, we deal with a conformal mapping of an ellipsoid to a sphere in which the selected meridian on the ellipsoid is mapped onto the sphere without distortions. Such a mapping allows us to interpret the Gauss-Krüger projection as a double mapping that has not been recognised so far.

Keywords: conformal mapping, double mapping, Gauss-Krüger projection, transverse Mercator projection

Zusammenfassung

Carl Friedrich Gauß ist in der Geodäsie für die Kartenprojektion – die sogenannte Gauß-Krüger-Abbildung – besonders bekannt. Sie wird manchmal auch transversale Mercator-Projektion genannt und in vielen Ländern der Welt amtlich verwendet. Gauß hat ebenfalls andere Varianten konformer Abbildungen eines Rotationsellipsoids auf eine Kugel sowie einer Kugel und eines Ellipsoids in eine Ebene untersucht. In dieser Arbeit beschäftigen wir uns mit einer konformen Abbildung eines Ellipsoids auf eine Kugel, bei der der ausgewählte Meridian auf dem Ellipsoid ohne Verzerrungen auf die Kugel abgebildet wird. Eine solche Abbildung erlaubt die Interpretation der Gauß-Krüger-Abbildung als eine bisher nicht bekannte Doppelabbildung.

Schlüsselwörter: *konforme Abbildung, Doppelabbildung, Gauß-Krüger-Projektion, transversale Mercator-Projektion*

1 Historical Background and Introduction

In an article on spheroidal Earth projections (1807), Mollweide explained the stereographic, Mercator and equal-area projections of an ellipsoid as double projections. First, the ellipsoid is mapped to a sphere in an appropriate way (conformally, or equivalently) and then the sphere to the plane. The mapping of an ellipsoid to a sphere is defined so that the meridians on the ellipsoid are mapped to the meridians on the sphere, $\lambda = \Lambda$, where λ is the longitude on the sphere and Λ is the longitude on the rotational ellipsoid. His derivations were based on the similarity or equality of elementary quadrilaterals. For the sake of more accurate calculation at that time, he developed the derived formulas into series.

Gauss used conformal mapping to transfer points from the Earth's ellipsoid to the plane. The task – to map one surface on another so that the image was similar to the original in the smallest details – was first mentioned in Gauss's letter to Schumacher of 5 July 1816; at the same time this task was recommended as the subject of a prize question for the academy.

Gauss sent a solution to the Scientific Society of Copenhagen at the end of 1822. This Society, at Schumacher's urging, had twice published a prize question. Gauss's solution was first published in 1825 in Schumacher's astronomical treatises (Krüger 1912).

As can be seen from his geodetic legacy, Werke Band 9, (Gauß 1903a–e), Gauss investigated various conformal mappings of the Earth's ellipsoids for geodetic purposes between 1816 and 1820. These included in particular the conformal double projection of the ellipsoid onto a sphere and then to a plane, one using a stereographic projection and the other using a transverse Mercator projection.

In an article on the general solution to the problem of mapping part of a given surface to another given surface so that the mapped area was in the smallest details similar to the original (1825), Gauss gave examples of conformal mapping of an ellipsoid into a plane and onto a sphere. For a conformal mapping of an ellipsoid, his general assumption was that the meridians on the ellipsoid were mapped to the meridians on the sphere ($\lambda = \Lambda$). He developed this notion further and it was published in his legacy in 1903, in the 9th volume, entitled *Conforme Doppelprojection des Sphäroids auf die Kugel und die Ebene*. In *Untersuchungen über Gegenstände der Höheren Geodäsie. Erste Abhandlung* in 1843, Gauss again dealt with the conformal mapping of an ellipsoid to a sphere with another generalization that allowed a better adaptation of a given area on the ellipsoid ($\lambda = \alpha\Lambda$).

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In his article on mappings of the Earth's ellipsoid (1891), Hammer first gave a historical account of determining the dimensions of the Earth's ellipsoid. He then showed Mollweide's approach and, based on the development in series with the sines of the double angles, gave tables for the Bessel and Clarke's 1866 ellipsoid. Hammer proved that Mollweide's conformal mapping was very close to the central projection of the ellipsoid onto the sphere. He noted that such a mapping is not equidistant and that a distinction should be made between *Vergrösserungsverhältniss and Längenverhältniss*. He then tackled a conformal mapping of an ellipsoid to a sphere that allowed a better fit of a given area of the ellipsoid ($\lambda = \alpha\Lambda$). He mentioned Gauss's condition for determining parameter α and gave tables and instructions for calculations with the "Gauss sphere".

In a book (1897) and a series of articles (1899–1900), Schreiber described in detail the double conformal projection of the Prussian state survey. He defined the projection of an ellipsoid on a sphere as a conformal projection that mapped parallels to parallels. The consequence of this was that the meridians had to be mapped into meridians, hence Schreiber concluded that $\lambda = \alpha\Lambda$ can be taken.

An account of conformal mappings of a rotational ellipsoid on a sphere can be found in the third volume of the well-known geodetic manual *Handbuch der Vermessungskunde*, which was first printed in 1872 (Jordan et al. 1923). The ninth chapter of this manual, covering 28 pages, describes in great detail a conformal mapping of an ellipsoid to a sphere *(Konforme Abbildung des Ellipsoids auf die Kugel)*. The introductory part cites the works of Gauss (1843), Schreiber (1897, 1899–1900) and Krüger (1914). It includes basic formulas, choice of constants, series developments, azimuth reduction, auxiliary tables and numerical examples. It is a conformal mapping using a Gauss sphere ($\lambda = \alpha\Lambda$).

Since the 10th edition in 1956, the *Handbuch der Vermessungskunde* has been revised in terms of volume and contents. The chapter on the conformal mapping of an ellipsoid to a sphere has been shortened to ten pages and is now found in the second part of the fourth volume (Jordan et al. 1959). A conformal mapping using a Gauss sphere is also described ($\lambda = \alpha\Lambda$).

An important person in the history of the Gauss-Krüger projection was the Bulgarian geodesist W. Hristow. With the help of several theorems from differential geometry and function theory, Hristow achieved a particularly concise and transparent presentation by using isometric or isothermal coordinate systems on the ellipsoid. He first published his thoughts in the *Zeitschrift für Vermessungswesen* and later summarized them (Hristow 1943, 1955). Many other authors also wrote about the Gauss-Krüger projection, such as Boltz (1943), König and Weise (1951), Großmann (1976), Ecker (1978), Schödlbauer (1981), Krack (1981), Heitz (1985), Heck (1987), Klotz (1993) and, more recently, Bretterbauer (2003), Schur (2005), Panasiuk et al. (2009), Bian and Li (2012) and Masaharu (2017).

All the authors who have dealt with Gauss and his contribution to geodetic cartography distinguish between his two approaches: double mapping of an ellipsoid by using a sphere as an intermediate surface into a plane, and direct mapping of an ellipsoid into a plane. Double mapping always involves a conformal mapping of an ellipsoid to a sphere using only a Gauss sphere ($\lambda = \alpha\Lambda$).

Schödlbauer (1981, 1982) started with series of rectangular coordinates of the Gauss-Krüger projection according to Boltz (1943) and Hristow (1955). He noticed that some members of the series could be singled out and interpreted together as the mapping of the ellipsoid to a sphere. The mapping of the ellipsoid to the sphere was not conformal because the latitude and longitude on the ellipsoid and on the sphere were identified. Thus, it was not a question of double mapping an ellipsoid onto a sphere and then a sphere into a plane, but of interpreting the Gauss-Krüger projection as the sum of two mappings: the mapping of a sphere formed by neglecting the flattening of the ellipsoid and the addition of the remaining terms so that the sum eventually gave a Gauss-Krüger projection. The underlying idea of replacing the spherical components contained in the power series with strictly closed formulas was also applied to the inverse problem (Krack 1981), dealt with by Schödlbauer.

A paper by Panasiuk et al. (2009) deals with conformal mapping from the plane of the Gauss-Krüger projection onto a sphere without distortion at the central meridian of the mapping region. The paper is not easy to understand because it is written in very poor English, but since it contains a multitude of mathematical formulas, it is not impossible to follow. The authors investigate the image of an ellipsoidal network of meridians and parallels on a sphere and analyse the distortions. The Gauss-Krüger projection is not treated as a composition of mapping an ellipsoid to a sphere and then a sphere to a plane.

In an earlier paper (Lapaine 2021), I showed that a direct conformal mapping of an ellipsoid into a plane, known as a Gauss-Krüger projection, could also be interpreted as a double mapping. However, this is not an application of the Gauss sphere but a mapping to a sphere of the image obtained by the Gauss-Krüger projection by using inverse mapping of the transverse Mercator projection of the sphere. The explanation lies in the fact that the composition of conformal mappings is again a conformal mapping.

In this paper, the same result is obtained but in a different way. Starting from an ellipsoid that is mapped to a sphere, provided that the selected meridian on the ellipsoid is mapped without distortion to the corresponding meridian on the sphere, an analytical extension of that map is applied to obtain a conformal map of the whole ellipsoid to the sphere.

2 The Sphere and Isometric Latitude

The equation of a sphere is usually written in parametric form with geographic parameterization

$$X = R\cos\varphi\cos\lambda \ , \ Y = R\cos\varphi\sin\lambda \ , \ Z = R\sin\varphi \ , \tag{1}$$

where $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\lambda \in [-\pi, \pi)$ are geographic latitude and longitude, respectively, and $R > 0$ is the radius. The geographic coordinates φ and λ are not isometric coordinates. However, to apply the properties of the functions of a complex variable, we need isometric coordinates. Therefore, we will define the conformal or isometric latitude q as

$$dq = \frac{d\varphi}{\cos\varphi} \ , \tag{2}$$

whence, by integration, with the choice of the addition constant zero, we obtain (Großmann, 1976):

$$q = \ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = \frac{1}{2}\ln\frac{1+\sin\varphi}{1-\sin\varphi} = \tanh^{-1}(\sin\varphi) \ . \tag{3}$$

3 Conformal Mapping of a Rotational Ellipsoid to a Sphere so that one Meridian is Mapped without Distortions

The equation of a rotational ellipsoid is usually written in parametric form with geodetic parameterization

$$X = N(\Phi)\cos\Phi\cos\Lambda \ , \ Y = N(\Phi)\cos\Phi\sin\Lambda \ ,$$
$$Z = N(\Phi)(1-e^2)\sin\Phi \ , \tag{4}$$

where $\Phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\Lambda \in [-\pi, \pi)$ are geodetic latitude and longitude, respectively, $a > b > 0$ are semi-axes of the ellipsoid, $e = \sqrt{\frac{a^2-b^2}{a^2}}$ and $N(\Phi)$ is the radius of curvature of the intersection along the first vertical

$$N(\Phi) = \frac{a}{\sqrt{1-e^2\sin^2\Phi}} \ . \tag{5}$$

The geodetic coordinates Φ and Λ are not isometric coordinates. However, to apply the properties of the functions of a complex variable, we need isometric coordinates. The isometric latitude Q on the ellipsoid is defined by

$$dQ = \frac{M(\Phi)d\Phi}{N(\Phi)\cos\Phi} \ , \tag{6}$$

whence by integration, with an additive constant of zero, we obtain (Großmann 1976):

$$Q(\Phi) = \tanh^{-1}(\sin\Phi) - e\tanh^{-1}(e\sin\Phi) \ . \tag{7}$$

Whenever we mention an ellipsoid in this paper, we mean a rotational ellipsoid.

The arc length of any meridian on a rotational ellipsoid from the equator to the point corresponding to the geodetic latitude Φ is determined by the formula

$$X(\Phi) = \int_0^{\Phi} M(\Phi)d\Phi \tag{8}$$

where $M(\Phi)$ is the radius of curvature of the meridian

$$M(\Phi) = \frac{a(1-e^2)}{\sqrt{(1-e^2\sin^2\Phi)^3}} \ . \tag{9}$$

It is known that the integral in (8) cannot be expressed by means of elementary functions in closed form, so developments in the series are usually applied. For instance, using the approach by König and Weise (1951), we can get:

$$X(\Phi) = A(\Phi + b_1\sin 2\Phi + b_2\sin 4\Phi + b_3\sin 6\Phi + b_4\sin 8\Phi + \ldots) \tag{10}$$

where

$$A = a(1-n)(1-n^2)\left(1 + \frac{9}{4}n^2 + \frac{225}{64}n^4\right) \tag{11}$$

$$b_1 = n\left(-\frac{3}{2} + \frac{9}{16}n^2 - \frac{537}{128}n^4\right)$$

$$b_2 = \frac{n^2}{2}\left(\frac{15}{8} - \frac{15}{16}n^2 + \frac{135}{1024}n^4\right)$$

$$b_3 = \frac{n^3}{3}\left(-\frac{35}{16} + \frac{315}{256}n^2\right)$$

$$b_4 = \frac{n^4}{4}\left(\frac{315}{128} - \frac{189}{128}n^2\right) .$$

The coefficients in (11) are expressed using the third flatness

$$n = \frac{a-b}{a+b} \ . \tag{12}$$

The terms of order n^7 have been omitted in (11). We can get from (10) the length of the meridian arc from the equator to the north pole for

$$\Phi = \frac{\pi}{2}$$

$$X = A\frac{\pi}{2} \ . \tag{13}$$

The arc length of any meridian on a sphere of radius R from the equator to the point corresponding to latitude φ is determined by the formula

$$X(\varphi) = R\varphi \ . \tag{14}$$

The length of the arc of the meridian on the sphere from the equator to the north pole is

$$X = R\frac{\pi}{2} \ . \tag{15}$$

If the length of the meridian on the ellipsoid is equal to the length of the meridian on the sphere, then by comparing (13) and (15), it must be

$$R = A \ . \tag{16}$$

Hence the name rectifying radius for A. Furthermore, if we want the length of the meridian arc on the ellipsoid from the equator to any point with geodetic latitude Φ to be equal to the length of the meridian arc on the sphere from the equator to the point with latitude φ, it should be

$$A\left(\Phi + b_1 \sin 2\Phi + b_2 \sin 4\Phi + b_3 \sin 6\Phi + b_4 \sin 8\Phi + \ldots\right) = R\varphi \tag{17}$$

that is, because (16)

$$\Phi + b_1 \sin 2\Phi + b_2 \sin 4\Phi + b_3 \sin 6\Phi + b_4 \sin 8\Phi + \ldots = \varphi . \tag{18}$$

To get a conformal mapping of the ellipsoid to the sphere for which (18) holds true, we first express Φ in (18) using the isometric latitude Q on the ellipsoid according to (7), and φ on the sphere using the isometric latitude q on the sphere according to (3):

$$\Phi(Q) + b_1 \sin 2\Phi(Q) + b_2 \sin 4\Phi(Q) + \ldots = \sin^{-1}(\tanh q). \tag{19}$$

Let us analytically extend (19)

$$\begin{aligned}&\Phi(Q+i\Lambda) + b_1 \sin 2\Phi(Q+i\Lambda) + b_2 \sin 4\Phi(Q+i\Lambda) + \ldots \\ &= \sin^{-1}\left(\tanh(q+i\lambda)\right) \ .\end{aligned} \tag{20}$$

Relation (20) shows a conformal mapping of an ellipsoid to a sphere in which (19) is valid along the central meridian of the mapping area ($\Lambda = \lambda = 0$) and both (18) and (17) are valid, that is, the length of the meridian arc on the ellipsoid from the equator to any point with geodetic latitude Φ will be equal to the length of the meridian arc on the sphere from the equator to the point with latitude φ. If we manage to express q and λ from equation (19), our problem will be solved, because for the given geodetic coordinates Φ and Λ on the ellipsoid we will be able to calculate the corresponding geographical coordinates φ and λ on the sphere. This will be shown in the following.

Let us denote

$$\Phi(Q+i\Lambda) = \xi + i\eta \ . \tag{21}$$

Furthermore, let us analytically extend (7)

$$Q + i\Lambda = \tanh^{-1}\left(\sin(\xi+i\eta)\right) - e\tanh^{-1}\left(e\sin(\xi+i\eta)\right) \tag{22}$$

with $\xi = \Phi$ for $\eta = \Lambda = 0$.

For the given Φ we can get Q from (7). Then, for the given Q and Λ, ξ and η should be calculated from (22). This is not possible directly because (22) is an irrational equation. However, we will first transform (22) to

$$\sin(\xi+i\eta) = \tanh\left[Q + i\Lambda + e\tanh^{-1}\left(e\sin(\xi+i\eta)\right)\right] \tag{23}$$

and then iterate. We can choose the initial value by putting $e = 0$ in (23). The procedure described converges rapidly for a small e. Once the iteration is complete, we finally have $z = \sin(\xi+i\eta)$ and then

$$\xi + i\eta = \sin^{-1} z \ . \tag{24}$$

The procedure described is feasible if we have software that can operate with complex numbers. Otherwise, instead of iterating expression (23), we can transform it into a system of two equations with real numbers by decomposing it into real and imaginary parts, and solve it again with an iterative approach (see the Appendix for details):

$$\xi = \tan^{-1}\frac{\sinh\left[Q + \frac{e}{2}\tanh^{-1}\frac{2e\sin\xi\cosh\eta}{1+e^2\left(\sinh^2\eta + \sin^2\xi\right)}\right]}{\cos\left[\Lambda + \frac{e}{2}\tan^{-1}\frac{2e\cos\xi\sinh\eta}{1-e^2\left(\sinh^2\eta + \sin^2\xi\right)}\right]}$$

$$\eta = \tanh^{-1}\frac{\sin\left[\Lambda + \frac{e}{2}\tan^{-1}\frac{2e\cos\xi\sinh\eta}{1-e^2\left(\sinh^2\eta + \sin^2\xi\right)}\right]}{\cosh\left[Q + \frac{e}{2}\tanh^{-1}\frac{2e\sin\xi\cosh\eta}{1+e^2\left(\sinh^2\eta + \sin^2\xi\right)}\right]} \ . \tag{25}$$

Practical calculations have shown that two iterations are sufficient even for a wider zone. Based on (20) and (21) we can now write

$$\begin{aligned}&\sin^{-1}\left(\tanh(q+i\lambda)\right) \\ &= \xi + i\eta + b_1 \sin 2(\xi+i\eta) + b_2 \sin 4(\xi+i\eta) + \ldots\end{aligned} \tag{26}$$

If we denote

$$\sin^{-1}\left(\tanh(q+i\lambda)\right) = u + iv \tag{27}$$

we have

$$u + iv = \xi + i\eta + b_1 \sin 2(\xi+i\eta) + b_2 \sin 4(\xi+i\eta) + \ldots \ . \tag{28}$$

From there, by decomposing into real and imaginary parts, we get

$$u=\xi+b_1\sin 2\xi\cosh 2\eta+b_2\sin 4\xi\cosh 4\eta+\ldots$$

$$v=\eta+b_1\cos 2\xi\sinh 2\eta+b_2\cos 4\xi\sinh 4\eta+\ldots\ . \quad (29)$$

It remains to express q and λ from (27) using u and v.

From (27) we first have

$$\tanh(q+i\lambda)=\sin(u+iv) \quad (30)$$

from where by decomposing into real and imaginary parts (see (A1)–(A9) in the Appendix) and changing Q for q, Λ for λ, ξ for u and η for v) we have

$$\sin\varphi=\tanh q=\frac{\sin u}{\cosh v}\ ,\ \tan\lambda=\frac{\sinh v}{\cos u}\ , \quad (31)$$

$$\varphi=\sin^{-1}\frac{\sin u}{\cosh v}\ ,\ \lambda=\tan^{-1}\frac{\sinh v}{\cos u}\ . \quad (32)$$

Thus, a conformal mapping of the ellipsoid to the sphere that preserves the lengths on the selected meridian is defined. Note that in the previous derivations, without reducing the generality, and for the sake of writing formulas more simply, the initial meridian $\lambda = 0$ was chosen. For any other meridian corresponding to longitude λ_0, $\lambda - \lambda_0$ should be used instead of λ in all formulas.

4 Gauss-Krüger Projection of a Rotational Ellipsoid as a Double Mapping

Having defined in the previous section a conformal mapping of an ellipsoid onto a sphere that maps the selected meridian without distortion, we can explain the Gauss-Krüger projection (arrow 1 in Fig. 1) as a double mapping. First, the ellipsoid is conformally mapped to a sphere according to the expressions from the previous section (arrow 2 in Fig. 1), and then that sphere is mapped to the plane (arrow 3 in Fig. 1) according to the known equations for the transverse Mercator projection of the sphere:

$$x=R\tan^{-1}\frac{\sinh q}{\cos\lambda}=R\tan^{-1}\frac{\tan\varphi}{\cos\lambda} \quad (33)$$

$$y=R\tanh^{-1}\frac{\sin\lambda}{\cosh q}=R\tanh^{-1}(\sin\lambda\cos\varphi)\ . \quad (34)$$

Let us note that

$$x=Ru=Au=A(\xi+b_1\sin 2\xi\cosh 2\eta+b_2\sin 4\xi\cosh 4\eta+\ldots) \quad (35)$$

$$y=Rv=Av=A(\eta+b_1\cos 2x\sinh 2\eta+b_2\cos 4x\sinh 4\eta+\ldots), \quad (36)$$

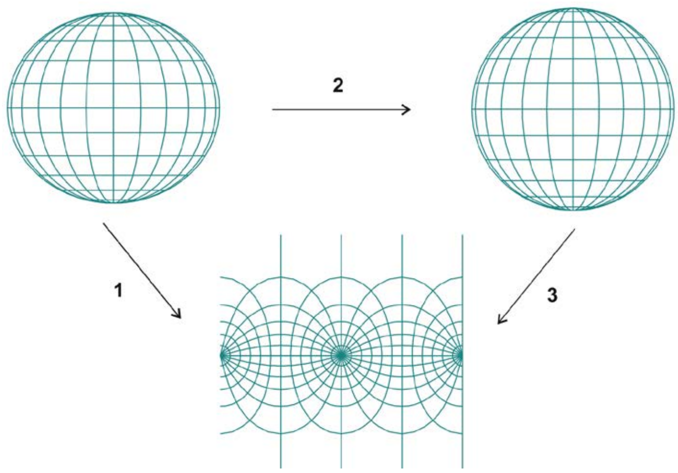

Fig. 1: (1) Gauss-Krüger projection of an ellipsoid into a plane; (2) Conformal projection of an ellipsoid on a sphere; (3) Transverse Mercator projection of a sphere into a plane

that is, in the derivation of mapping an ellipsoid to a sphere, we actually had these formulas (eqs. (29)) up to the proportionality factor A.

The x and y coordinates are the unreduced coordinates. If we need reduced coordinates, let us call them N and E and, as it is usual in geodesy and cartography, use the expressions

$$N=m_0x \quad (37)$$

$$E=m_0y+K \quad (38)$$

where m_0 is the linear scale factor along the central meridian, and K is false easting.

The following are three numerical examples that will enable a comparison of the results obtained according to the formulas in this paper with those in Schödlbauer (1981), Bretterbauer (2003) and Schur (2005).

Example 1

Schödlbauer (1981) gives two numerical examples for calculating Gauss-Krüger coordinates.

In the first example, the ellipsoid is Bessel's (a = 6 377 397.155 m, b = 6 356 078.962822 m). According to (11) A = 6 366 742.520236 m. A point was selected with geodetic coordinates

Φ = 50°51′18.3891″, Λ = 1°59′43.1538″.

The coordinates on the sphere, which is a conformal image of this ellipsoid and has an equidistantly mapped central meridian, according to formulas (32), are

φ = 50°42′51.0500″, λ = 1°59′47.9573″.

Table 1 gives the unreduced coordinates in the Gauss-Krüger projection.

Tab. 1: Comparison of unreduced Gauss-Krüger coordinates

	x [m]	y [m]
Schödlbauer (1981, Rechenbeispiel 1)	5 637 286.050	140 479.772
This paper (35)–(36)	5 637 286.049	140 479.772

Table 1 shows that the difference between the numerical values of the coordinates in the Schödlbauer paper (1981, Rechenbeispiel 1) and the values calculated according to the formulas in this paper is not larger than 1 mm.

Example 2

In his second example, Schödlbauer (1981) uses the International Ellipsoid ($a = 6\,378\,388$ m, $b = 6\,356\,911.946$ m). According to (11) $A = 6\,367\,654.499994$ m. A point was selected with geodetic coordinates

$\Phi = 50°41'09.4140''$,
$\Lambda = 7°09'06.9400'' - 9° = -1°50'53.0600''$.

The coordinates on the sphere, which is a conformal image of this ellipsoid and has an equidistantly mapped central meridian, according to formulas (32) are

$\varphi = 50°32'37.7681''$, $\lambda = -1°50'57.5741''$.

Table 2 gives the reduced coordinates in the Gauss-Krüger projection with $m_0 = 0.9996$ and $K = 500\,000$ m.

Table 2 shows that the difference between the numerical values of the coordinates in the Schödlbauer paper (1981, Rechenbeispiel 2) and the values calculated according to the formulas in this paper is again not larger than 1 mm.

Tab. 2: Comparison of reduced Gauss-Krüger coordinates

	N [m]	E [m]
Schödlbauer (1981, Rechenbeispiel 2)	5 616 645.735	369 446.254
This paper (37)–(38)	5 616 645.734	369 446.254

Example 3

The third example refers to the Bessel ellipsoid ($a = 6\,377\,397.155$ m, $b = 6\,356\,078.962822$ m) and is mentioned by Bretterbauer (2003) and Schur (2005), who calculated the coordinates according to the formulas of Krüger (1912) and Klotz (1993). According to (11) $A = 6\,366\,742.520236$ m. A point was selected with geodetic coordinates

$\Phi = 48°$, $\Lambda = 50'$.

The coordinates on the sphere, which is a conformal image of this ellipsoid and has an equidistantly mapped central meridian, according to formulas (32) are

$\varphi = 47°52'26.2776''$, $\lambda = 50°01'43.4047''$.

Table 3 gives the unreduced coordinates in the Gauss-Krüger projection. It shows that the difference between the numerical values of the coordinates in Bretterbauer (2003) and Schur (2005) and the values calculated according to the formulas in this paper is not larger than 1 mm, even in the wider zone.

Tab. 3: Comparison of unreduced Gauss-Krüger coordinates

	x [m]	y [m]
Bretterbauer (2003)	6 649 901.177	3 617 710.791
Schur after Krüger (2005)	6 649 901.176041	3 617 710.791658
Schur after Klotz (2005)	6 649 901.176674	3 617 710.791314
This paper (35)–(36)	6 649 901.176592	3 617 710.791269

5 Conclusion

In addition to the conformal mapping of an ellipsoid into a plane, known today as the Gauss-Krüger projection, Gauss also carried out research into other variants of conformal mappings of a rotational ellipsoid onto a sphere and a sphere and an ellipsoid into a plane.

Up to now the Gauss sphere has been mentioned and used in geodetic calculations and geodetic cartography. It is a sphere that in a certain way adapts well to an ellipsoid in a certain area. The Gauss-Krüger projection is a conformal mapping of an ellipsoid into a plane that maps a selected meridian without distortion.

The main contribution of this paper is of a theoretical nature. It shows that there is a sphere on which the ellipsoid can be conformally mapped so that one of the meridians is mapped without distortions. This was done by analytically extending the mapping of the selected meridian to the entire ellipsoid.

The consequence is that the Gauss-Krüger projection can be interpreted as a double mapping: a conformal mapping of the ellipsoid to a sphere and a conformal mapping of the sphere to a plane. At the same time, an important property is preserved – the selected meridian is mapped without distortions.

The accuracy of the coordinates of the points in the plane of the Gauss-Krüger projection is equal to the accuracy of calculating the length of the meridian arc.

Appendix

We will show how equation (24) can be transformed into system (25) in the similar way as in (Lapaine 1996). Let us first consider the simpler case and suppose that

$$Q + i\Lambda = \tanh^{-1}\left(\sin\left(\xi + i\eta\right)\right) . \tag{A1}$$

We want to express Q and Λ from (A1). Let us first note that it can be written

$$\tanh^{-1}\left(\sin\left(\xi+i\eta\right)\right) = \frac{1}{2}\ln\frac{1+\sin\left(\xi+i\eta\right)}{1-\sin\left(\xi+i\eta\right)} = \ln\tan\left(\frac{\pi}{4}+\frac{\xi+i\eta}{2}\right)$$
$$= \ln\frac{\sin\left(\frac{\pi}{2}+\xi\right)+i\sinh\eta}{\cos\left(\frac{\pi}{2}+\xi\right)+\cosh\eta} = \ln\frac{\cos\xi + i\sinh\eta}{\cosh\eta - \sin\xi} . \tag{A2}$$

Let us denote, (with the e base of natural logarithms)

$$z = \rho e^{i\psi} = \frac{\cos\xi + i\sinh\eta}{\cosh\eta - \sin\xi} . \tag{A3}$$

Now we can calculate

$$\rho^2 = \frac{\cos^2\xi + \sinh^2\eta}{\left(\cosh\eta - \sin\xi\right)^2} = \frac{\cosh^2\eta - \sin^2\xi}{\left(\cosh\eta - \sin\xi\right)^2} = \frac{\cosh\eta + \sin\xi}{\cosh\eta - \sin\xi} \tag{A4}$$

and

$$\tan\psi = \frac{\sinh\eta}{\cos\xi} . \tag{A5}$$

Given the definition of a complex logarithmic function

$$\ln z = \ln\rho + i\psi \tag{A6}$$

it follows that it is

$$\ln z = \frac{1}{2}\ln\frac{\cosh\eta + \sin\xi}{\cosh\eta - \sin\xi} + i\tan^{-1}\frac{\sinh\eta}{\cos\xi} . \tag{A7}$$

Finally, based on (A2)–(A7) we can write

$$Q = \frac{1}{2}\ln\frac{\cosh\eta + \sin\xi}{\cosh\eta - \sin\xi} = \frac{1}{2}\ln\frac{1+\frac{\sin\xi}{\cosh\eta}}{1-\frac{\sin\xi}{\cosh\eta}} = \tanh^{-1}\frac{\sin\xi}{\cosh\eta} \tag{A8}$$

$$\Lambda = \tan^{-1}\frac{\sinh\eta}{\cos\xi} . \tag{A9}$$

Suppose now that it is

$$Q + i\Lambda = \tanh^{-1}\left(\sin\left(\xi+i\eta\right)\right) - e\tanh^{-1}\left(e\sin\left(\xi+i\eta\right)\right) \tag{A10}$$

We want to express Q and Λ from (A10). For this purpose, we first introduce the complex variable $m + in$ as follows

$$\sin\left(\xi+i\eta\right) = \tanh\left(m+in\right) . \tag{A11}$$

If (A11) is compared with (A1) it can be seen that the obtained results can be used, but Q should be replaced with m and Λ with n.

So, it is

$$\tanh m = \frac{\sin\xi}{\cosh\eta} , \ \tan n = \frac{\sinh\eta}{\cos\xi} . \tag{A12}$$

If it were still possible to introduce the complex variable $p + ir$ so that it was

$$e\sin\left(\xi+i\eta\right) = \tanh\left(p+ir\right) \tag{A13}$$

then the solution would be obtained in a simple form

$$Q + i\Lambda = m + in - e\left(p+ir\right) = m - ep + i\left(n - er\right) . \tag{A14}$$

Let us show how p and r can be determined. Let us first introduce ξ' and η' in this way:

$$\sin\left(\xi'+i\eta'\right) = e\sin\left(\xi+i\eta\right) . \tag{A15}$$

Now we can use the above expressions again using the model of (A12)

$$\tanh p = \frac{\sin\xi'}{\cosh\eta'} , \ \tan r = \frac{\sinh\eta'}{\cos\xi'} . \tag{A16}$$

A return to ξ and η remains. For this purpose, we first note that (A15) by applying addition formulas and separating the real and imaginary part can be written in the form

$$\sin\xi'\cosh\eta' = e\sin\xi\cosh\eta$$
$$\cos\xi'\sinh\eta' = e\cos\xi\sinh\eta . \tag{A17}$$

Then using (A16) we write

$$\tanh 2p = \frac{2\tanh p}{1+\tanh^2 p} = \frac{2\sin\xi'\cosh\eta'}{\cosh^2\eta' + \sin^2\xi'}$$

$$\tan 2r = \frac{2\tan r}{1-\tan^2 r} = \frac{2\cos\xi'\sinh\eta'}{\cos^2\xi' - \sinh^2\eta'} . \tag{A18}$$

We already have the expressions in the numerator in equations (A18) according to (A17). For expressions in the denominator, relations (A17) need to be squared and summed, and then applying the basic formulas of ordinary and hyperbolic trigonometry reduced to

$$\sin^2\xi' + \cosh^2\eta' = 1 + e^2\left(\sin^2\xi + \sinh^2\eta\right)$$
$$\cos^2\xi' - \sinh^2\eta' = 1 - e^2\left(\sin^2\xi + \sinh^2\eta\right) . \tag{A19}$$

Therefore, (A18) can now be written in the form

$$\tanh 2p = \frac{2e\sin\xi\cosh\eta}{1+e^2\left(\sin^2\xi+\sinh^2\eta\right)}$$
$$\tan 2r = \frac{2e\cos\xi\sinh\eta}{1-e^2\left(\sin^2\xi+\sinh^2\eta\right)} . \tag{A20}$$

In this way we have managed to separate the real and imaginary parts in (A10), and then based on (A14) and derived relations we finally have

$$Q = \tanh^{-1}\frac{\sin\xi}{\cosh\eta} - \frac{e}{2}\tanh^{-1}\frac{2e\sin\xi\cosh\eta}{1+e^2\left(\sin^2\xi+\sinh^2\eta\right)}$$
$$\Lambda = \tan^{-1}\frac{\sinh\eta}{\cos\xi} - \frac{e}{2}\tan^{-1}\frac{2e\cos\xi\sinh\eta}{1-e^2\left(\sin^2\xi+\sinh^2\eta\right)} . \tag{A21}$$

If ξ and η are to be calculated for the given Q and Λ, then we must reduce (A21) to a form suitable for the iterative procedure. Namely, system (A21) is a system of irrational equations that cannot be solved by any direct method. From (A21) we have

$$\tanh^{-1}\frac{\sin\xi}{\cosh\eta} = Q + \frac{e}{2}\tanh^{-1}\frac{2e\sin\xi\cosh\eta}{1+e^2\left(\sin^2\xi+\sinh^2\eta\right)}$$
$$\tan^{-1}\frac{\sinh\eta}{\cos\xi} = \Lambda + \frac{e}{2}\tan^{-1}\frac{2e\cos\xi\sinh\eta}{1-e^2\left(\sin^2\xi+\sinh^2\eta\right)} . \tag{A22}$$

and then

$$\frac{\sin\xi}{\cosh\eta} = \tanh\left[Q + \frac{e}{2}\tanh^{-1}\frac{2e\sin\xi\cosh\eta}{1+e^2\left(\sin^2\xi+\sinh^2\eta\right)}\right]$$
$$\frac{\sinh\eta}{\cos\xi} = \tan\left[\Lambda + \frac{e}{2}\tan^{-1}\frac{2e\cos\xi\sinh\eta}{1-e^2\left(\sin^2\xi+\sinh^2\eta\right)}\right] \tag{A23}$$

and thence

$$\tan\xi = \frac{\sinh\left[Q + \frac{e}{2}\tanh^{-1}\frac{2e\sin\xi\cosh\eta}{1+e^2\left(\sin^2\xi+\sinh^2\eta\right)}\right]}{\cos\left[\Lambda + \frac{e}{2}\tan^{-1}\frac{2e\cos\xi\sinh\eta}{1-e^2\left(\sin^2\xi+\sinh^2\eta\right)}\right]}$$

$$\tanh\eta = \frac{\sin\left[\Lambda + \frac{e}{2}\tan^{-1}\frac{2e\cos\xi\sinh\eta}{1-e^2\left(\sin^2\xi+\sinh^2\eta\right)}\right]}{\cosh\left[Q + \frac{e}{2}\tanh^{-1}\frac{2e\sin\xi\cosh\eta}{1+e^2\left(\sin^2\xi+\sinh^2\eta\right)}\right]} \tag{A24}$$

and finally

$$\xi = \tan^{-1}\frac{\sinh\left[Q + \frac{e}{2}\tanh^{-1}\frac{2e\sin\xi\cosh\eta}{1+e^2\left(\sinh^2\eta+\sin^2\xi\right)}\right]}{\cos\left[\Lambda + \frac{e}{2}\tan^{-1}\frac{2e\cos\xi\sinh\eta}{1-e^2\left(\sinh^2\eta+\sin^2\xi\right)}\right]}$$

$$\eta = \tanh^{-1}\frac{\sin\left[\Lambda + \frac{e}{2}\tan^{-1}\frac{2e\cos\xi\sinh\eta}{1-e^2\left(\sinh^2\eta+\sin^2\xi\right)}\right]}{\cosh\left[Q + \frac{e}{2}\tanh^{-1}\frac{2e\sin\xi\cosh\eta}{1+e^2\left(\sinh^2\eta+\sin^2\xi\right)}\right]} \tag{A25}$$

which is the system (25). Thus, we have shown how the equation (24) leads to the system (25).

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References

Bian, Sh.-F., Li, H.-P. (2012): Mathematical Analysis in Cartography by Means of Computer Algebra System. In: Bateira, C. (Ed.): Cartography – A Tool for Spatial Analysis. IntechOpen. DOI: 10.5772/50159.

Boltz, H. (1943): Formeln und Tafeln zur numerischen (nicht logarithmischen) Berechnung Gauß-Krüger'schen Koordinaten aus den geographischen Koordinaten. Veröffentlichung des Geodätischen Instituts Potsdam, Neue Folge Nr. 111, Potsdam.

Bretterbauer, K. (2003): Gebrauchsformeln für die UTM-Projektion nach Krüger. Österreichische Zeitschrift für Vermessung und Geoinformation, 91, 3, 163–165.

Ecker, E. (1978): Conformal Mappings of the Earth Ellipsoid. manuscripta geodetica, No. 2/3, 229–251.

Gauß, C. F. (1825): Allgemeine Auflösung der Aufgabe: Die Theile einer gegebenen Fläche auf einer andern gegebenen Fläche so abzubilden, dass die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird. Astronomische Abhandlungen, herausgegeben von H. C. Schumacher, 3. Heft, Altona, 5–30.

Gauß, C. F. (1843–1845): Untersuchungen über Gegenstände der Höheren Geodäsie. Erste Abhandlung. (23. Oktober 1843). Abhandlungen der Mathematischen Classe der Königlichen Gesellschaft der Wissenschaften in Göttingen 2, 1845, 3–34 (also in: Gauß: Werke. Band 4, 261–290).

Gauß, C. F. (1903a): Conforme Doppelprojection des Sphäroids auf die Kugel und die Ebene, Nachlass, Gauß: Werke. Band 9, 105–116. Herausgegeben von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. In Commission bei B. G. Teubner in Leipzig.

Gauß, C.F. (1903b): Stereographische Projection der Kugel auf die Ebene, Nachlass, Gauß: Werke. Band 9, 117–122. Herausgegeben von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. In Commission bei B.G. Teubner in Leipzig.

Gauß, C.F. (1903c): Übertragung der Kugel auf die Ebene durch Mercators Projection, Nachlass, Gauß: Werke. Band 9, 123–134. Herausgegeben von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. In Commission bei B.G. Teubner in Leipzig.

Gauß, C.F. (1903d): Conforme Übertragung des Sphäroids auf den Kegelmantel, Nachlass, Gauß: Werke. Band 9, 135–140. Herausgegeben von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. In Commission bei B.G. Teubner in Leipzig.

Gauß, C.F. (1903e): Conforme Abbildung des Sphäroids in der Ebene. (Projectionsmethode der Hannoverschen Landesvermessung.) Nachlass, Gauß: Werke. Band 9, 141–204. Briefwechsel, Über die Formeln für die hannoversche Landesvermessung, Band 9, 205–218. Herausgegeben von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. In Commission bei B.G. Teubner in Leipzig.

Großmann, W. (1976): Geodätische Rechnungen und Abbildungen in der Landesvermessung. 3. Aufl., Wittwer, Stuttgart.

Hammer, E. (1891): Zur Abbildung des Erdellipsoids. Zeitschrift für Vermessungswesen, vol. 20, no. 22, 609–617; no. 24, 641–661.

Heck, B. (2003): Rechenverfahren und Auswertemodelle der Landesvermessung. 3. Aufl., Wichmann Verlag, Karlsruhe.

Heitz, S. (1988): Coordinates in Geodesy. Springer-Verlag, Berlin etc.

Hristow, W.K. (1943): Die Gauß-Krüger'schen Koordinaten auf dem Ellipsoid. Verlag B.G. Teubner, Leipzig und Berlin.

Hristow, W.K. (1955): Die Gaußschen und geographischen Koordinaten auf dem Ellipsoid von Krassowsky. VEB Verlag Technik, Berlin.

Jordan, W., Reinhertz, C., Eggert, O. (1923): Handbuch der Vermessungskunde. Dritter Band: Landesvermessung und Grundaufgaben der Erdmessung. 7. erweiterte Auflage, Stuttgart.

Jordan, W., Eggert, O., Kneissl, M. (1959): Handbuch der Vermessungskunde. Band IV: Mathematische Geodäsie (Landesvermessung), Zweite Hälfte, Die geodätische Berechnungen auf der Kugel und auf dem Ellipsoid. 10., völlig neu bearbeitete und gegliederte Ausgabe, Stuttgart.

Klotz, J. (1993): Eine analytische Lösung der Gauß-Krüger-Abbildung. Zeitschrift für Vermessungswesen, Heft 3, 106–116.

König, R., Weise, K.H. (1951): Mathematische Grundlagen der höheren Geodäsie und Kartographie. 1. Band, Springer-Verlag, Berlin.

Krack, K. (1981): Die Umwandlung von Gaußsche konformen Koordinaten in geographischen Koordinaten des Bezugsellipsoiden auf der Grundlage des transversalen Mercatorentwurfes. Allgemeine Vermessungs-Nachrichten, Heft 5, 173–178.

Krüger, L. (1912): Konforme Abbildung des Erdellipsoids in der Ebene. Königlich Preußisches Geodätisches Institut, Veröffentlichung, Neue Folge, No. 52. Potsdam.

Krüger, L. (1914): Transformation der Koordinaten bei der konformen Doppelprojektion des Erdellipsoids auf die Kugel und die Ebene. Königlich Preußisches Geodätisches Institut, Veröffentlichung, Neue Folge, No. 60. Potsdam.

Lapaine, M. (1996): Preslikavanja u teoriji kartografskih projekcija (in Croatian). PhD Thesis, University of Zagreb, Faculty of Geodesy.

Lapaine, M. (2021): Gauss-Krügerova projekcija kao dvostruko preslikavanje (in Croatian). Geodetski list, 2021, 2, 85–102.

Masaharu, H. (2017): Development in understanding of Gauss-Krüger projection and its outcomes. Proceedings of the International Cartographic Association, 1, 1–4.

Mollweide, C.B. (1807): Einige Projectionsarten der sphäroidischen Erde. Zach's Monatliche Correspondenz zur Beförderung der Erd- und Himmels-Kunde, v. 16, Sept., 197–210.

Panasiuk, J., Balcerzak, J., Pędzich, P. (2009): New Approach to the Gauss-Krüger Projection of an Ellipsoid onto a Sphere, Proceedings of the 24th International Cartographic Conference, Chile, Non-referred paper. https://icaci.org/files/documents/ICC_proceedings/ICC2009/html/nonref/10.html.

Schödlbauer, A. (1981): Gaußsche konforme Abbildung von Bezugsellipsoiden in die Ebene auf der Grundlage des transversalen Mercatorentwurfs. Allgemeine Vermessungs-Nachrichten, Heft 5, 165–173.

Schödlbauer, A. (1982): Rechenformeln und Rechenbeispiele zur Landesvermessung. Teil 2, Wichmann-Skripte Heft 2, Herbert Wichmann Verlag Karlsruhe.

Schreiber, O. (1897): Die konforme Doppelprojektion der trigonometrischen Abteilung der königl. Preussischen Landesaufnahme. Berlin.

Schreiber, O. (1899–1900): Zur konformen Doppelprojection der preußischen Landesaufnahme. Sphäroid und Kugel. Gaußsche Projection. Zeitschrift für Vermessungswesen, vol. 28 (1899), 491–502, 593–613; vol. 29 (1900), no. 11, 257–281; no. 12, 289–310.

Schur, P. (2005): Vergleich zweier Berechnungsverfahren zur Gauß-Krüger Abbildung breiter Meridianstreifen. Österreichische Zeitschrift für Vermessung und Geoinformation, 93, 1, 45–47.

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