

Uncertainty of Results of Laser Scanning Data with Correlated Systematic Effects by Monte Carlo Methods

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Summary

The uncertainty caused by a systematic effect in measurements is determined. The systematic effect can be detected from a long series of repetitions. The series is split up into blocks of equal number of observations. Significant changes of the mean values of the measurements between the blocks indicate a systematic effect. Significant jumps between the standard deviations of the data of the blocks give the standard deviation of the systematic effect. To decide whether a systematic effect is present in the observations of a block, the autocorrelations of the time series formed by the repetitions are estimated. The autocorrelations have to go to zero, otherwise a systematic effect must be suspected. The method is applied to the repeated measurements of the three-dimensional coordinates of points of a grid by a laser scanner. The covariances of the measurements are estimated in a special multivariate linear model and the covariances of the systematic effects by the auto- and cross-covariances of the time series of the coordinates. The causes of the systematic effects are discussed. The uncertainty and the expanded uncertainty of the sum of the distances of the laser scanner to the grid points is computed by Monte Carlo methods. The multivariate normal distribution is assumed for the measured coordinates and the multivariate rectangular distribution for their systematic effects. Random variates for the latter distribution are generated by Monte Carlo methods. It turns out that the uncertainty and the expanded uncertainty of the sum of distances increase if correlated observations are introduced instead of independent ones. The uncertainty increases again if systematic effects are added.

Zusammenfassung

Die Unsicherheit wird bestimmt, die ein systematischer Effekt in Messungen hervorruft. Der systematische Effekt lässt sich durch eine lange Serie von Wiederholungen aufdecken. Die Serie wird in Blöcke mit gleicher Anzahl von Beobachtungen zerlegt. Signifikante Änderungen der Mittelwerte der Messungen zwischen den Blöcken deuten auf einen systematischen Effekt hin. Signifikante Sprünge der Standardabweichungen der Daten ergeben die Standardabweichung des systematischen Effektes. Um zu entscheiden, ob ein systematischer Effekt in den Beobachtungen eines Blockes präsent ist, werden die Autokorrelationen der Zeitreihe geschätzt, die durch die Wiederholungen gebildet wird. Die Autokorrelationen müssen gegen Null gehen, anderenfalls ist ein systematischer Effekt zu vermuten. Die Methode wird auf die wiederholten Messungen der dreidimensionalen Koordinaten der Punkte eines Gitters durch einen Laserscanner angewendet. Die Kovarianzen der Messungen werden in einem speziellen multivariaten linearen Modell geschätzt und die Kovarianzen der systematischen Effekte durch

die Auto- und Kreuzkorrelationen der Zeitreihen der Koordinaten. Die Ursachen der systematischen Effekte werden diskutiert. Die Unsicherheit und die erweiterte Unsicherheit der Summe der Entfernungen des Laserscanners von den Gitterpunkten werden durch Monte-Carlo-Methoden berechnet. Die multivariate Normalverteilung wird für die gemessenen Koordinaten und die multivariate Rechteckverteilung für ihre systematischen Effekte angenommen. Zufallszahlen für die letztere Verteilung werden durch Monte-Carlo-Methoden generiert. Es stellt sich heraus, dass die Unsicherheit und die erweiterte Unsicherheit der Summe der Entfernungen wächst, wenn anstelle unabhängiger Beobachtungen korrelierte Messungen eingeführt werden. Die Unsicherheit erhöht sich weiter, wenn systematische Effekte hinzugefügt werden.

1 Introduction

Even after calibrating a measuring instrument, its observations might be distorted by a systematic effect. A systematic effect in measurements is therefore characterized by being neither random nor observable (Lira and Wöger 2006). Its influence on the uncertainty of measurements cannot be treated by traditional or conventional statistics, which is not founded on Bayes' theorem. The »Guide to the Expression of Uncertainty in Measurement (GUM)« (ISO 1995) however, shows how to handle a systematic effect. GUM is now considered as international standard for determining the uncertainty in measurements, cf. Sommer and Siebert (2004), Cox and Siebert (2006). GUM groups the components of uncertainty according to their method of evaluation into a Type A and a Type B component. Type A is determined by statistical methods, Type B by experience and knowledge about the measuring instrument. Both types may contain random and systematic effects. Although not explicitly stated, GUM applies Bayesian statistics to treat the Type B component of uncertainty, cf. Weise and Wöger (1993), Kacker and Jones (2003). Systematic effects are defined as random variables with probability density functions so that their uncertainties are expressed by variances and covariances.

This is obviously difficult to accept by someone not familiar with Bayesian statistics. One can even read that GUM cannot be recommended because of this definition (Schmidt 2003). It is reasoned that if a systematic effect is a random variable, it cannot be eliminated by the measuring process. However, unknown quantities or unknown parameters are introduced as random variables

in the Bayesian approach since probability is not only associated with random events but more generally with statements. Not the systematic effect is random, it is the incomplete knowledge about the value of the effect for which a probability is introduced resulting from a probability density function, cf. Koch (2007, p. 34).

Although a systematic effect is not observable, the uncertainty in measurements caused by the systematic effect can be determined. This will be shown here by repeated observations under laboratory conditions where influences of the environment do not affect the measurements. A long series of repetitions is divided into blocks of equal numbers of measurements for which the mean values together with their confidence intervals, called coverage intervals by GUM, are computed. If the mean value of one block lies outside the confidence interval of a different block, a systematic effect can be suspected. The mean values of the standard deviations of the blocks of data together with their confidence intervals are also computed. If the standard deviation of one block lies outside the confidence interval of the standard deviation of a different block, the difference of the variances can be attributed to a systematic effect.

To derive the uncertainty of results derived from the measurements, a block of data has to be found not distorted or only slightly distorted by a systematic effect. The correlations of the data can be used to find such a block. Measurements taken in a rapid succession, for instance by a laser scanner, tend to be correlated. Conditions, which affect the measuring process, generally change slowly. The same conditions therefore prevail for some time, thus causing correlations. The repeated measurements mentioned above give time series from which correlations can be estimated in a special multivariate linear model (Koch 1999, p. 250). Also autocorrelations can be obtained by estimating autocovariance functions (Koch et al. 2010). The correlations are not affected by a constant systematic effect. But if a systematic effect varies over a long series of repetitions, the correlations get distorted. The autocorrelations should decrease with the increase of the time differences between the measurements because they stop depending on each other. If the decrease does not happen, a systematic effect has to be suspected. A block of the repeated observations with no systematic or a small systematic effect is therefore characterized by possessing a small standard deviation of the measurements and by autocorrelations which go to zero with increasing time differences.

Generally one does not know, how a systematic effect distorts the measurements. The effect is therefore simply expressed by an additive bias, cf. Kacker and Lawrence (2007). We will introduce a vector of observations and a vector of additive systematic effects. As mentioned above, the observations are correlated which has to be taken care of when determining the uncertainties, cf. Sommer and Siebert (2006), Koch (2008a). If the measurements are correlated, the systematic effects will be also correlated.

Measurements of the laser scanner Leica HDS 3000 of the Institute of Geodesy and Geoinformation (University of Bonn) are analyzed. This instrument measures the three-dimensional coordinates of points on the surface of objects. It is applied, for instance, in reverse engineering in order to fit to the measured coordinates free-form surfaces or time depending surfaces (Koch 2010). The coordinates are defined in a local coordinate system, the origin of which is located in the center of the instrument. The x -axis lies horizontally, the y -axis coincides with the center of the lines of sight of the instrument and the z -axis points to the zenith. The three measured coordinates of a point result from an observed distance, a horizontal and a vertical angle. The instrument scans an object for a fixed value x in the direction of the positive z -axis. The value x is then increased for the next scan so that a grid of points is formed. The three coordinates of a point are highly correlated, since a change in distance may cause changes in all three coordinates with the same sign. Also the measurements of the coordinates of different points are correlated, because they are taken in a rapid sequence.

The variances and the covariances of the measured x -, y - and z -coordinates of each point and the covariances of the coordinates of each pair of points are estimated from the repeated observations in a special multivariate linear model mentioned above. The repeated measurements can also be used to build up time series for the x -, y - and z -coordinates. This can be justified by the measuring process of a laser scanner, where the x -coordinate is mainly determined by the horizontal angle, the y -coordinate by the distance and the z -coordinate by the vertical angle. We assume that the observations of the x -coordinates in the sequence of the scans establish a time series which does not stop with the first repetition but continues with each repetition. The same is assumed for the y - and z -coordinates. Auto- and cross-correlations then follow from the estimated auto- and cross-covariance functions of the three coordinates.

The coordinates of points of a grid on a perpendicularly standing board are measured by the laser scanner with many repetitions. To determine the effect of the correlations of the observations plus their systematic effects on the uncertainty of derived results, the measurements are nonlinearly transformed to the sum of the distances of the laser scanner to the points of the grid. This sum is very sensitive to correlations, cf. Koch (2008a), Koch and Kuhlmann (2009). The uncertainty and the expanded uncertainty, expressed by the standard deviation and the confidence interval, of the sum are determined by Monte Carlo methods, cf. Koch (2007, p. 216), JCGM (2008). The multivariate normal distribution with the expected values, variances and covariances estimated in the multivariate model is assumed for the measurements.

GUM discusses the rectangular or uniform, the trapezoidal and the triangular distribution for the Type B component of uncertainty. Generalizations such as the asymmetric trapezoidal distribution (Kacker and Lawrence

2007) or the isocurvilinear trapezoidal distribution (Kacker and Lawrence 2009) have been derived. These distributions are univariate distributions. For the correlated systematic effects, they have to be generalized to multivariate distributions which are not available. However, random samples from a multivariate rectangular distribution in the interval [0,1] having a prescribed correlation matrix can be generated by Monte Carlo methods due to Falk (1999), Gentle (2003, p. 207). These samples are transformed to random variates from a multivariate rectangular distribution in any interval and to a multivariate trapezoidal or triangular distribution with a prescribed correlation matrix by the inversion method, cf. Koch (2007, p. 194). Numerical studies have shown that these multivariate distributions have the same property as the multivariate normal distribution: the marginal distributions of the multivariate distributions have the same form as the univariate distributions (Koch 2008a). The multivariate rectangular distribution with expected values zero, with widths determined by the standard deviations of the systematic effects and with the prescribed correlation matrix from the estimated auto- and cross-correlations is used to determine the uncertainty caused by the correlated systematic effects. The estimated auto- and cross-correlations represent the dependencies of the x -, y - and z -coordinates so that they adequately express the correlations of the systematic effects.

The paper is organized as follows. Section 2 introduces the special linear multivariate model for estimating the variances and covariances. In addition the method of estimating the auto- and cross-covariances is presented. Section 3 deals with the Monte Carlo methods for determining the uncertainty and expanded uncertainty of quantities derived from the measurements. Section 4 gives the results for the analysis of the measurements of the laser scanner Leica HDS 3000 and points out the systematic effects and their causes. The paper finishes with conclusions.

2 Estimation of variances and covariances

Let the three-dimensional coordinates x_l, y_l, z_l with $l \in \{1, \dots, n_g\}$ of a grid with n_c columns and n_r rows of $n_g = n_c \times n_r$ points be measured, for instance, by a laser scanner. It scans one column after the other starting at the lower left corner and stopping at the upper right corner. As mentioned in the introduction, the measurements of the three coordinates of each point but also the coordinates of different points have to be considered as dependent. To estimate the $n_k \times n_k$ covariance matrix Σ of the coordinates with $n_k = 3n_g$, the measurements are repeated and divided into blocks each with n_w observations.

The covariance matrix Σ is estimated in a special multivariate model derived from the general multivariate model, cf. Anderson (1958, p. 179), Koch (1999,

p. 238), by introducing - instead of a vector of unknown parameters - only one unknown parameter β_i with $i \in \{1, \dots, n_k\}$, which is the coordinate i of the n_k coordinates of the n_g grid points. Let $\mathbf{o}_i = (o_{wi})$, $w \in \{1, \dots, n_w\}$, $i \in \{1, \dots, n_k\}$ be the vector of n_w repeated measurements of the coordinate i . The estimate $\hat{\beta}_i$ of the unknown parameter β_i then follows under the condition $n_k \leq n_w - 1$ from the multivariate model by, cf. Koch (2008a), Koch et al. (2010),

$$\hat{\beta}_i = \frac{1}{n_w} \sum_{w=1}^{n_w} o_{wi}$$

for $i \in \{1, \dots, n_k\}$ (1)

and the elements $\hat{\sigma}_{ij}$ of the estimate $\hat{\Sigma}$ of the $n_k \times n_k$ covariance matrix Σ of the coordinates by

$$\hat{\sigma}_{ij} = \frac{1}{n_w - 1} \sum_{w=1}^{n_w} (\hat{\beta}_i - o_{wi})(\hat{\beta}_j - o_{wj})$$

for $i, j \in \{1, \dots, n_k\}$ (2)

with

$$\hat{\Sigma} = (\hat{\sigma}_{ij}) . (3)$$

The covariance matrix $\hat{\Sigma}$ is positive definite in case of normally distributed observations. If the measurements o_{wi} are distorted by a constant systematic effect, the constant cancels, as can be seen from (1) and (2) so that the estimated variances and covariances are not affected.

The covariance matrix $\hat{\Sigma}$ estimated by (3) gives the covariances of the three coordinates of all grid point. As mentioned in the introduction, we also consider only the x -, y - and z -coordinates instead of the individual coordinates of each grid point. By collecting the $n_s = n_g \times n_w$ values of the x -coordinates of the vectors $\mathbf{o}_i = (o_{wi})$, $w \in \{1, \dots, n_w\}$, $i \in \{1, \dots, n_k\}$ from (1) we find

$$\begin{matrix} o_{11}, & o_{14}, & o_{17}, & \dots, & o_{1,n_k-2}, \\ o_{21}, & o_{24}, & o_{27}, & \dots, & o_{2,n_k-2}, \\ \dots & \dots & \dots & \dots & \dots \\ o_{n_w,1}, & o_{n_w,4}, & o_{n_w,7}, & \dots, & o_{n_w,n_k-2}. \end{matrix} (4)$$

Each column of (4) refers to the repeated measurements of the x -coordinate of one grid point. The estimate \hat{x} of the x -coordinates of the n_g grid points follows as mean value from the estimates by (1), which shall be denoted after renumbering by \bar{x}_l for $l \in \{1, \dots, n_g\}$. Thus,

$$\hat{x} = \frac{1}{n_g} \sum_{l=1}^{n_g} \bar{x}_l . (5)$$

The variance $\hat{\sigma}_{\hat{x}}^2$ of the mean is obtained by

$$\hat{\sigma}_{\hat{x}}^2 = \frac{1}{(n_g - 1)n_g} \sum_{l=1}^{n_g} (\bar{x}_l - \hat{x})^2 . (6)$$

The $1 - \alpha$ confidence interval for x follows from the t -distribution by, cf. Koch (2007, p. 113),

$$P(\hat{x} - \hat{\sigma}_{\hat{x}} t_{1-\alpha; n_g-1} < x < \hat{x} + \hat{\sigma}_{\hat{x}} t_{1-\alpha; n_g-1}) = 1 - \alpha (7)$$

with $t_{1-\alpha; n_g-1} = (F_{1-\alpha; 1, n_g-1})^{1/2}$ and $F_{1-\alpha; 1, n_g-1}$ being the $(1 - \alpha)$ -percentage point of the F-distribution.

If the point null hypothesis is tested

$$H_0 : x = x_0 \quad \text{versus} \quad H_1 : x \neq x_0, \quad (8)$$

where x_0 is a given value, the null hypothesis is rejected in case x_0 lies outside the confidence interval (7), cf. Koch (2007, p. 83). The value x_0 then differs significantly from x . Accordingly, \hat{y} and \hat{z} and the confidence intervals and hypothesis tests are obtained. Significant changes of \hat{x} , \hat{y} and \hat{z} between the blocks of data point out systematic effects, which are not constant but vary during the repeated observations.

The variance σ_x^2 of x is estimated by $\hat{\sigma}_x^2$ as mean value of the estimates by (2), which shall be denoted by $\hat{\sigma}_{\bar{x}_l}^2$ for $l \in \{1, \dots, n_g\}$. We obtain

$$\hat{\sigma}_x^2 = \frac{1}{n_g} \sum_{l=1}^{n_g} \hat{\sigma}_{\bar{x}_l}^2 \quad (9)$$

and the standard deviation $\hat{\sigma}_x = (\hat{\sigma}_x^2)^{1/2}$. The variance $\hat{\sigma}_{\hat{\sigma}_x^2}^2$ of $\hat{\sigma}_x^2$ follows corresponding to (6) by

$$\hat{\sigma}_{\hat{\sigma}_x^2}^2 = \frac{1}{(n_g - 1)n_g} \sum_{l=1}^{n_g} (\hat{\sigma}_{\bar{x}_l}^2 - \hat{\sigma}_x^2)^2. \quad (10)$$

The $1 - \alpha$ confidence interval for σ_x^2 expressed as standard deviation σ_x is obtained corresponding to (7) by

$$P((\hat{\sigma}_x^2 - \hat{\sigma}_{\hat{\sigma}_x^2} t_{1-\alpha; n_g-1})^{1/2} < \sigma_x < (\hat{\sigma}_x^2 + \hat{\sigma}_{\hat{\sigma}_x^2} t_{1-\alpha; n_g-1})^{1/2}) = 1 - \alpha \quad (11)$$

and the hypothesis test according to (8). Likewise, $\hat{\sigma}_y^2$ and $\hat{\sigma}_z^2$ with their confidence intervals and hypothesis tests are found.

The estimates $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ are used to determine the standard deviations of the systematic effects in the coordinates. Let $\hat{\sigma}_{xmi}$ be the minimum standard deviation for x of the blocks of data and $\hat{\sigma}_{xma}$ be the maximum standard deviation of the blocks. Let both values differ significantly. The standard deviation $\hat{\sigma}_{xe}$ of the systematic effect in the coordinate x is then computed by

$$\hat{\sigma}_{xe} = (\hat{\sigma}_{xma}^2 - \hat{\sigma}_{xmi}^2)^{1/2} \quad (12)$$

and $\hat{\sigma}_{ye}$ and $\hat{\sigma}_{ze}$ accordingly.

As mentioned in the introduction, we assume that the measurements of the x -coordinates of the grid points in the sequence of the scans establish a time series which does not stop with the first repetition of the measurements but continues with each repetition and ends with the n_w th repetition. A corresponding time series is built up for the y -coordinate and a third time series for the z -coordinate. As can be seen from (4), the time series contain coordinates with different mean values estimated by $\bar{x}_l, \bar{y}_l, \bar{z}_l$ and used in (5). The values are subtracted to obtain time series with means equal to zero. By calling $x(n)$

the realisations for the time series of the x -coordinates and renumbering them from 1 to $n_s = n_g \times n_w$, we get

$$x(n) \quad \text{for} \quad n \in \{1, \dots, n_s\} \quad (13)$$

and accordingly the time series $y(n)$ and $z(n)$ for the y - and z -coordinates.

We assume that these three time series represent stationary processes which have constant expected values and moment functions which are only dependent on the time difference τ , cf. Priestley (1981, p. 104), Koch and Schmidt (1994, p. 166). The values of τ refer to the indices n of the time series $x(n), y(n)$ and $z(n)$. To estimate the auto- and cross-covariance functions for x, y, z , only one time series for each coordinate is available. These time series therefore have to be ergodic in order to replace the ensemble average by the time average. Ergodicity with respect to the expected value and to the autocovariance function exists if the values for the autocovariance function tend to zero for large values of τ , cf. Papoulis (1977, p. 356), Koch and Schmidt (1994, p. 213). By replacing the expected values of the three time series by their means, which are equal to zero as mentioned above, the estimate $\hat{C}_{xx}(\tau)$ of the autocovariance function for x follows by

$$\hat{C}_{xx}(\tau) = \frac{1}{n_s} \sum_{n=1}^{n_s-\tau} x(n)x(n+\tau)$$

$$\text{for } n \in \{1, \dots, n_s\}, \tau \in \{0, \dots, n_s - 1\}. \quad (14)$$

This estimate is biased because of dividing by n_s instead of $n_s - \tau$. It is preferred, however, to the unbiased one which gets large variances for τ approaching $n_s - 1$. Accordingly, $\hat{C}_{yy}(\tau)$ and $\hat{C}_{zz}(\tau)$ are obtained. The estimated autocovariance functions are positive semidefinite functions since their Fourier transforms give the periodograms which are non-negative, cf. Priestley (1981, p. 399), Koch and Schmidt (1994, p. 221).

The estimated autocorrelation function $\hat{\rho}_{xx}(\tau)$ follows with

$$\hat{\rho}_{xx}(\tau) = \hat{C}_{xx}(\tau) / \hat{C}_{xx}(0) \quad (15)$$

and $\hat{\rho}_{yy}(\tau)$ and $\hat{\rho}_{zz}(\tau)$ accordingly. With increasing time differences, the dependency between the coordinates should decrease so that $\hat{\rho}_{xx}(\tau), \hat{\rho}_{yy}(\tau)$ and $\hat{\rho}_{zz}(\tau)$ should go to zero. If they do not, the time series are not ergodic and systematic effects have to be suspected.

The estimate $\hat{C}_{xy}(\tau)$ of the cross-covariance function of x and y follows with

$$\hat{C}_{xy}(\tau) = \frac{1}{n_s} \sum_{n=1}^{n_s-\tau} x(n)y(n+\tau)$$

$$\text{for } n \in \{1, \dots, n_s\}, \tau \in \{0, \dots, n_s - 1\} \quad (16)$$

and $\hat{C}_{xz}(\tau)$ and $\hat{C}_{yz}(\tau)$ accordingly. The estimated cross-correlation function $\hat{\rho}_{xy}(\tau)$ is obtained with

$$\hat{\rho}_{xy}(\tau) = \hat{C}_{xy}(\tau) / (\hat{C}_{xx}(0)\hat{C}_{yy}(0))^{1/2} \quad (17)$$

and $\hat{\rho}_{xz}(\tau)$ and $\hat{\rho}_{yz}(\tau)$ accordingly. It was mentioned in connection with (3) that the estimated variances and covariances are independent of a constant systematic effect. The same holds true for the auto- and cross-covariance functions estimated by (14) and (16).

By means of the 3×3 submatrices $\hat{\mathbf{R}}(\tau)$

$$\hat{\mathbf{R}}(\tau) = \begin{pmatrix} \hat{\rho}_{xx}(\tau) & \hat{\rho}_{xy}(\tau) & \hat{\rho}_{xz}(\tau) \\ & \hat{\rho}_{yy}(\tau) & \hat{\rho}_{yz}(\tau) \\ & & \hat{\rho}_{zz}(\tau) \end{pmatrix} \quad (18)$$

for $\tau \in \{0, \dots, n_g - 1\}$

the $n_k \times n_k = 3n_g \times 3n_g$ matrix $\hat{\mathbf{R}}_e$ of correlations is built up with

$$\hat{\mathbf{R}}_e = \begin{pmatrix} \hat{\mathbf{R}}(0) & \hat{\mathbf{R}}(1) & \dots & \hat{\mathbf{R}}(n_g - 1) \\ & \hat{\mathbf{R}}(0) & \dots & \hat{\mathbf{R}}(n_g - 2) \\ & & \dots & \dots \\ & & & \hat{\mathbf{R}}(0) \end{pmatrix}. \quad (19)$$

The correlations in (19) are representative for the dependency of the x -, y - and z -coordinates. They are used as correlations for the systematic effects of the measured coordinates. The covariance matrix $\hat{\mathbf{C}}_e$ of the systematic effects follows from (19) by multiplying the rows and columns of the submatrices $\hat{\mathbf{R}}(\tau)$ in (18) by the estimated standard deviations $\hat{\sigma}_{xe}$, $\hat{\sigma}_{ye}$ and $\hat{\sigma}_{ze}$ of the systematic effects from (12). It is checked numerically by computing the Cholesky factorization of the covariance matrix $\hat{\mathbf{C}}_e$ whether it is positive definite. This factorization is needed for generating random values for the Monte Carlo method.

3 Monte Carlo method

GUM expresses a mathematical model of a measurement by the functional relationship

$$Y = f(X_1, \dots, X_N) \quad (20)$$

where Y is the output quantity, the so-called measurand which is intended to be measured, and X_1, \dots, X_N are N random variables, the input quantities. We generalize (20) and replace the measurand Y by an $n \times 1$ random vector \mathbf{y} of measurands and the function f by an $n \times 1$ vector \mathbf{f} of functions and obtain with the $N \times 1$ input vector $\mathbf{x} = |X_1, \dots, X_N|^T$ instead of (20)

$$\mathbf{y} = \mathbf{f}(\mathbf{x}). \quad (21)$$

Let $p(\mathbf{x})$ be the probability density function of the vector \mathbf{x} , the expected value $E(\mathbf{y})$ of the vector \mathbf{y} is then defined by, cf. Koch (2007, p. 39),

$$E(\mathbf{y}) = \int_{\mathcal{X}} \mathbf{f}(\mathbf{x}) p(\mathbf{x}) d\mathbf{x} \quad (22)$$

with \mathcal{X} being the space containing the set of vectors \mathbf{x} with all possible values for \mathbf{x} . It shows that only the probability density function $p(\mathbf{x})$ for the input vector \mathbf{x} is

needed for the Monte Carlo estimate of $E(\mathbf{y})$. If m random samples \mathbf{x}_i are drawn for the random vector \mathbf{x} from the distribution $p(\mathbf{x})$

$$\mathbf{x}_i = |x_{1i}, \dots, x_{Ni}|' \quad \text{with } i \in \{1, \dots, m\}, \quad (23)$$

the random variates \mathbf{y}_i with $i \in \{1, \dots, m\}$ for the vector \mathbf{y} therefore follow from (21) by

$$\mathbf{y}_i = \mathbf{f}(\mathbf{x}_i). \quad (24)$$

The Monte Carlo estimate $\hat{D}(\mathbf{y})$ of the covariance matrix $D(\mathbf{y})$ of the vector \mathbf{y} is obtained by, cf. Cox and Siebert (2006), Koch (2007, p. 225),

$$\hat{D}(\mathbf{y}) = \frac{1}{m} \sum_{i=1}^m \left(\mathbf{f}(\mathbf{x}_i) - \hat{E}(\mathbf{f}(\mathbf{x})) \right) \left(\mathbf{f}(\mathbf{x}_i) - \hat{E}(\mathbf{f}(\mathbf{x})) \right)^T \quad (25)$$

with the estimate $\hat{E}(\mathbf{f}(\mathbf{x}))$ of the expectation $E(\mathbf{y})$

$$\hat{E}(\mathbf{f}(\mathbf{x})) = \frac{1}{m} \sum_{i=1}^m \mathbf{f}(\mathbf{x}_i). \quad (26)$$

The estimate $\hat{D}(\mathbf{y})$ of the covariance matrix $D(\mathbf{y})$, which is called uncertainty matrix by GUM, Supplement 1 (JCGM 2008), gives the uncertainty of the vector \mathbf{y} of measurands.

GUM expresses the expanded uncertainty of a measurand as a coverage interval. It is computed here for the vector \mathbf{y} as Bayesian confidence region, which is a region of highest posterior density (H. P. D. region), cf. Koch (2007, p. 71),

$$P(\mathbf{y} \in \mathcal{Y}_B) = \int_{\mathcal{Y}_B} p(\mathbf{y}) d\mathbf{y} = 1 - \alpha \quad (27)$$

with

$$p(\mathbf{y}_1) \geq p(\mathbf{y}_2) \quad \text{for } \mathbf{y}_1 \in \mathcal{Y}_B, \mathbf{y}_2 \notin \mathcal{Y}_B \quad (28)$$

and with α being the probability. The density function $p(\mathbf{y})$ for \mathbf{y} is determined by the random variates \mathbf{y}_i and \mathcal{Y}_B denotes the confidence region which is a subspace of the space \mathcal{Y} . It contains the set of vectors \mathbf{y} with all possible values for \mathbf{y} . If the vector \mathbf{y} has only one component, (27) and (28) define the confidence interval or coverage interval for Y .

GUM, Supplement 1, recommends sorting the random variates for the measurand in increasing order and determining the coverage interval by counting the sorted variates at both ends. This can be explained by the fact that each random sample has the probability of $1/m$ (Koch 2007, p. 201). However, sorting a large number m of random variates takes time. A histogram, for which sorting is not needed, is used to compute the Bayesian confidence interval according to (27) and (28). The probability of Y lying within a cell of suitable width is determined by the relative frequency. The probability at both ends of the histogram is added such that (28) is fulfilled until the probability α is reached. For m random variates, $m - 1$ cells

are chosen here. Counting the sorted random variates or adding the relative frequencies will then give approximately the same result, because at both ends of the confidence interval not more than one random variate per cell can be expected for a large number m .

As mentioned in the introduction, it will be assumed that an $n \times 1$ vector \mathbf{x}_o of measurements is biased by an additive $n \times 1$ vector \mathbf{x}_e of systematic effects. Thus, we obtain instead of (21)

$$\mathbf{y} = \mathbf{x}_o + \mathbf{x}_e . \quad (29)$$

We assume \mathbf{x}_o and \mathbf{x}_e being independent with probability density functions $p(\mathbf{x}_o)$ and $p(\mathbf{x}_e)$. Let m random samples \mathbf{x}_{oi} be drawn from $p(\mathbf{x}_o)$ and m samples \mathbf{x}_{ei} from $p(\mathbf{x}_e)$. We then obtain from (24) and (29) the random samples \mathbf{y}_i for \mathbf{y} by

$$\mathbf{y}_i = \mathbf{x}_{oi} + \mathbf{x}_{ei} . \quad (30)$$

Uncertainties for the $u \times 1$ vector $\boldsymbol{\beta}$ of quantities derived from the measurands have to be determined. The vector $\boldsymbol{\beta}$ could be obtained by a $u \times 1$ vector \mathbf{g} of linear transformations, e.g. as estimate by least-squares, or of nonlinear transformations, e.g. as distances from measured coordinates,

$$\boldsymbol{\beta} = \mathbf{g}(\mathbf{y}) . \quad (31)$$

The random variates β_i for $\boldsymbol{\beta}$ follow with

$$\beta_i = \mathbf{g}(\mathbf{y}_i) \quad (32)$$

and for the special case (30) with

$$\beta_i = \mathbf{g}(\mathbf{x}_{oi} + \mathbf{x}_{ei}) . \quad (33)$$

The Monte Carlo estimate $\hat{D}(\boldsymbol{\beta})$ of the covariance matrix $D(\boldsymbol{\beta})$ and the Bayesian confidence region follow corresponding to (25) up to (28). Again, the probability density function for $\boldsymbol{\beta}$ is not needed because of the propagation of the distributions of the input quantities.

We will assume the multivariate normal distribution for the measurements \mathbf{x}_o with expectation $\hat{\boldsymbol{\mu}}$ and covariance matrix $\hat{\boldsymbol{\Sigma}}$ estimated by (1) and (3)

$$\mathbf{x}_o \sim N(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}}) . \quad (34)$$

The m random samples \mathbf{x}_{oi} are drawn from (34) by means of a Cholesky factorization of $\hat{\boldsymbol{\Sigma}}$, cf. Koch (2007, p. 197). The vector \mathbf{x}_e of systematic effects cannot be observed so that their expected values are assumed to be zero. The range of values, however, can be inferred from the standard deviations $\hat{\sigma}_{x_e}, \hat{\sigma}_{y_e}, \hat{\sigma}_{z_e}$ from (12). Since the values for the systematic effects are equally likely, the multivariate rectangular distribution R is assumed with zero expectations and with covariance matrix $\hat{\mathbf{C}}_e$ resulting from (19), thus

$$\mathbf{x}_e \sim R(\mathbf{0}, \hat{\mathbf{C}}_e) . \quad (35)$$

The m random variates x_{ei} are generated by the method given by Falk (1999), Gentle (2003, p. 207). Samples $\mathbf{x}_i = (x_{ji})$ for $i \in \{1, \dots, m\}, j \in \{1, \dots, N\}$ are first generated from the normal distribution $\mathbf{x} \sim N(\mathbf{0}, \hat{\mathbf{R}}_e)$ with the correlation matrix $\hat{\mathbf{R}}_e$ from (19). The distribution functions $F(x_{ji})$ for x_{ji} of the standard normal distribution are computed to obtain the random variates $f_i = |F(x_{1i}), \dots, F(x_{Ni})|^T$. The components u_{ji} of f_i

$$u_{ji} = F(x_{ji}) \text{ for } i \in \{1, \dots, m\}, j \in \{1, \dots, N\} \quad (36)$$

are random variates from the multivariate rectangular distribution in the interval $[0, 1]$ with correlation matrix $\hat{\mathbf{R}}'_e$ which is close to the prescribed correlation matrix $\hat{\mathbf{R}}_e$. If identity of the given matrix $\hat{\mathbf{R}}_e$ and the matrix $\hat{\mathbf{R}}'_e$ is asked for, one can manipulate the matrix which one starts with such that the generated matrix agrees with the prescribed matrix. However, this will not be necessary, because $\hat{\mathbf{R}}_e$ is obtained by an estimation.

If the random variable U is uniformly distributed in the interval $[0, 1]$, the random variable $X = F^{-1}(U)$ has the distribution function $F(x)$. We therefore use the inversion method, cf. Koch (2007, p. 194), to compute by means of the random variates u_{ji} from (36) with the rectangular distribution in $[0, 1]$ and the correlation matrix $\hat{\mathbf{R}}'_e$, the random variates of the multivariate rectangular distribution in any interval or the random variates of the multivariate trapezoidal and triangular distribution with correlation matrix $\hat{\mathbf{R}}'_e$. For the inverses of the distribution functions of the rectangular, the trapezoidal or the triangular distribution see for instance Kacker and Lawrence (2007), Koch (2008a).

4 Uncertainty of results derived from the measurements of a laser scanner

The coordinates of $n_g = 42$ points of a rectangular grid with $n_c = 7$ columns and $n_r = 6$ rows on a perpendicularly standing plane board are measured in a laboratory by the laser scanner Leica HDS 3000. The distances between the points of the grid are about 0.22 m. The x - and z -coordinates of the corner points of the grid in the coordinate system of the laser scanner are given in Fig. 1. The minimum y -coordinate which gives the shortest distance of the laser scanner to the board is about 11.26 m. The points are scanned from negative to positive values z with increasing values x . The coordinate system of the laser scanner is used and the distances to the points of the grid do not vary much because of the small values for x and z in comparison to y . An addition constant, distance depending corrections or corrections due to changing incidence angles, as applied by Koch and Kuhlmann (2009), therefore need not to be introduced for the measurements. Also small instrumental errors, like collimation and trunnion axis error or vertical circle index error do not affect the observations. Large errors do not exist because of the

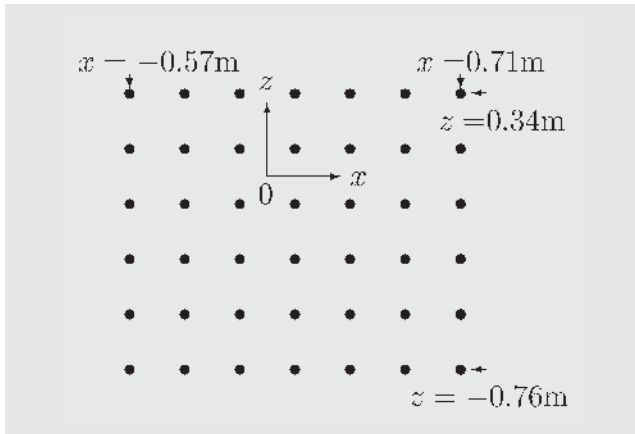


Fig. 1: Board with grid of 42 points and x-, z-coordinates of corner points

small standard deviations for the coordinates shown in the following.

The measurements of the three coordinates of the $n_g = 42$ grid points is continuously repeated up to 1320 times. The repetitions are divided into four blocks, each with 330 repetitions. These repetitions, i.e. $n_w = 330$, and every second repetition, i.e. $n_w = 165$, are used to estimate the $n_k \times n_k$ covariance matrix $\hat{\Sigma}$ of the $n_k = 3 \times 42 = 126$ coordinates of the grid points by (1) and (3). The condition $n_k \leq n_w - 1$ in (1) is fulfilled. The estimates $\hat{x}, \hat{y}, \hat{z}$ of the coordinates of the grid points by (5) and their 0.95% coverage intervals by (7) for the four blocks of repetitions are given in Tab. 1.

Tab. 1 shows that the estimates \hat{x} except for one and the estimates \hat{z} are identical. However, \hat{y} which represents the distance jumps by 0.9 mm. The value 11269.0 mm for repetitions 661-990 lies outside the coverage interval of 11269.9 mm for repetition 1-330 so that the hypothesis (8) that both values agree has to be rejected. The difference in \hat{y} therefore cannot be explained by the standard deviation $\hat{\sigma}_{\hat{y}}$ of \hat{y} used to

compute the coverage interval. The difference must be attributed to a systematic effect and can be interpreted as a correction for the distance measurements which varies over the extended series of repetitions. Choosing three blocks with 440 repetitions each show similar results.

The systematic effects cannot be determined by the jumps of \hat{y} shown in Tab. 1. However, standard deviations of the systematic effects can be inferred from the data. The estimates $\hat{\sigma}_x^2, \hat{\sigma}_y^2, \hat{\sigma}_z^2$ of the variances of the coordinates are therefore computed by (9) and expressed by standard deviations together with their coverage intervals from (11). The results are presented in Tab. 2. The estimate 0.12 mm for σ_x differs significantly from the estimate 0.10 mm judged by the coverage interval for 0.10 mm, 2.86 mm for σ_y differs significantly from 2.32 mm and 0.18 mm for σ_z from 0.15 mm. These differences can be attributed to systematic effects. The standard deviations of the systematic effects in the x-, y- and z-coordinates is therefore obtained from (12) by

$$\hat{\sigma}_{x_e} = 0.07 \text{ mm}, \hat{\sigma}_{y_e} = 1.67 \text{ mm}, \hat{\sigma}_{z_e} = 0.10 \text{ mm} . \quad (37)$$

The standard deviations $\hat{\sigma}_x$ and $\hat{\sigma}_z$ of Tab. 2 as well as $\hat{\sigma}_{x_e}$ and $\hat{\sigma}_{z_e}$ are smaller than $\hat{\sigma}_y$ and $\hat{\sigma}_{y_e}$ by more than a factor of 10. The standard deviations and the covariances of the y-coordinates and their systematic effects therefore predominantly influence the uncertainty of results derived from the measurements.

To find the block of 330 repetitions least influenced by systematic effects, a block with a small standard deviation $\hat{\sigma}_y$ has to be chosen. In addition, the estimates $\hat{\rho}_{yy}(\tau)$ of the autocorrelations for y from (15) with $n_s = n_g \times n_w = 13860$ in (14) have to decrease towards zero. To check it, the averaged sums of the absolute values of $\hat{\rho}_{yy}$ and also $\hat{\rho}_{xx}$ as well as $\hat{\rho}_{zz}$ are computed and given in Tab. 3 for $\tau = 20$ to $\tau = 41$ and for $\tau = 42$ to $\tau = 83$. With $\tau = 20$, the autocorrelations start to become small and $\tau = 41$

Tab. 1: Estimates of the x-, y-, z-coordinates with 0.95% coverage limits in mm

Repetition number	x-coordinate			y-coordinate			z-coordinate		
	lower l.	\hat{x}	upper l.	lower l.	\hat{y}	upper l.	lower l.	\hat{z}	upper l.
001-330	-52.3	83.1	218.5	11269.3	11269.9	11270.6	-324.1	-205.8	-87.6
331-660	-52.3	83.1	218.5	11269.0	11269.6	11270.3	-324.1	-205.8	-87.6
661-990	-52.3	83.1	218.4	11268.4	11269.0	11269.6	-324.1	-205.8	-87.6
991-1320	-52.3	83.0	218.4	11269.2	11269.8	11270.5	-324.1	-205.8	-87.6

Tab. 2: Estimates of the standard deviations $\sigma_x, \sigma_y, \sigma_z$ of the coordinates with 0.95% coverage limits in mm

Repetition number	standard deviation σ_x			standard deviation σ_y			standard deviation σ_z		
	lower l.	$\hat{\sigma}_x$	upper l.	lower l.	$\hat{\sigma}_y$	upper l.	lower l.	$\hat{\sigma}_z$	upper l.
001-330	0.09	0.11	0.12	2.46	2.49	2.52	0.12	0.16	0.18
331-660	0.09	0.11	0.12	2.55	2.60	2.64	0.12	0.16	0.18
661-990	0.10	0.12	0.13	2.82	2.86	2.89	0.14	0.18	0.21
991-1320	0.09	0.10	0.11	2.29	2.32	2.35	0.12	0.15	0.18

Tab. 3: Sums of absolute values of estimated autocorrelations $\hat{\rho}_{xx}, \hat{\rho}_{yy}, \hat{\rho}_{zz}$

Repetition number	Sums: $\tau = 20$ to $\tau = 41$ and $\tau = 42$ to $\tau = 83$					
	$\hat{\rho}_{xx}$		$\hat{\rho}_{yy}$		$\hat{\rho}_{zz}$	
001-330	0.047	0.052	0.011	0.007	0.067	0.078
331-660	0.016	0.021	0.026	0.020	0.076	0.088
661-990	0.010	0.008	0.010	0.006	0.074	0.087
991-1320	0.025	0.028	0.046	0.041	0.115	0.129

marks the end of the first scan of the $n_g = 42$ grid points by the laser scanner. The end of the second scan is reached with $\tau = 83$. The first and the third block show a fast decrease of $\hat{\rho}_{yy}$, but the first block has a smaller standard deviation $\hat{\sigma}_y$ than the third block as shown in Tab. 2. The first block with repetition 1-330 is therefore chosen as the block with possibly no or only small systematic effects for y . The uncertainty of the results of the measurements of this block are evaluated.

To demonstrate, how systematic effects change the autocorrelations, the values up to $\tau = 83$ of $\hat{\rho}_{yy,330}$ for the repetitions 1-330 with $n_s = 13\,860$, of $\hat{\rho}_{yy,660}$ for the repetitions 1-660 with $n_s = 27\,720$ and of $\hat{\rho}_{yy,1320}$ for the repetitions 1-1320 with $n_s = 55\,540$ are plotted in Fig. 2. The autocorrelations $\hat{\rho}_{yy,330}$ show positive and negative values which tend to zero, see also Fig. 3. The autocorrelations $\hat{\rho}_{yy,660}$ and $\hat{\rho}_{yy,1320}$ have positive values only which hardly decrease especially the last ones. With the increase of the number of repetitions, the autocorrelations become more and more distorted by systematic effects due to the changes of the mean values \hat{y} shown in Tab. 1.

Looking at $\hat{\rho}_{xx}$ and $\hat{\rho}_{zz}$ of Tab. 3, one recognizes that the autocorrelations do not decrease. This is confirmed by Fig. 3 where the autocorrelations $\hat{\rho}_{xx}, \hat{\rho}_{yy}$ and $\hat{\rho}_{zz}$ are plotted up to $\tau = 83$ for the repetitions 1-330. The values for $\hat{\rho}_{xx}$ are only positive and show peaks for $\tau \in \{6, 12, 18, \dots\}$ with the maximum at $\tau = 42$, i.e. for the time differences, which equal the time between the

scans of adjacent columns of the grid. The values for $\hat{\rho}_{zz}$ are positive and negative but have even more pronounced peaks at the same time differences. Slightly higher peaks for these differences appear when the repetitions 1-660 and 1-1320 are evaluated. Systematic effects for x and z are present. However, they will not affect the uncertainties of results derived from the systematic effects of the observations because of the small standard deviations $\hat{\sigma}_{xe}$ and $\hat{\sigma}_{ze}$ in comparison to $\hat{\sigma}_{ye}$ shown in (37).

As mentioned, a grid of points is scanned in the direction of the positive z -axis with increasing x values. To choose a grid, horizontal angles with identical increments for the columns of the grid and vertical angles with identical increments for the rows of the grid are set in advance. The measured coordinates then follow by the measured distances. When repeating a scan, the same horizontal and vertical angles are used to determine the coordinates by the measured distances. However, the horizontal and vertical angles cannot be exactly recovered for the repetitions. Judging from the results of Fig. 3, the same errors appear for the same vertical angles. They are not constant but change with time, thus causing high correlations for the z -coordinates with time differences equal to the time between the scans of adjacent columns of the grid. When recovering the horizontal angles for the repeated scans, the same errors appear for the same horizontal angles. They also drift with time and cause correlations for the x -coordinates of all time differences. Ob-

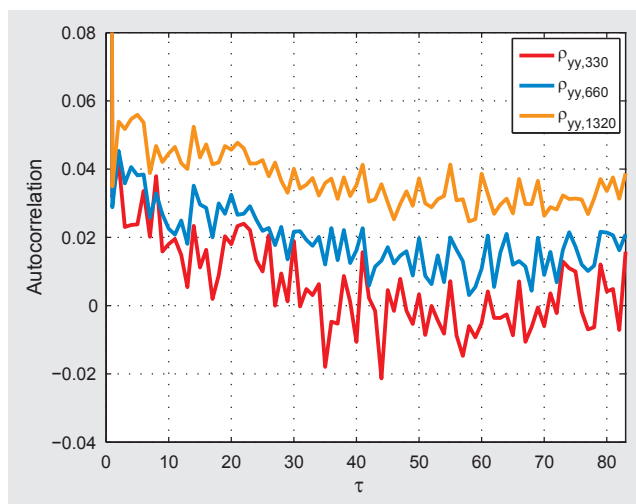


Fig. 2: Autocorrelations $\hat{\rho}_{yy}$ up to $\tau = 83$ for repetitions 1-330, 1-660 and 1-1320

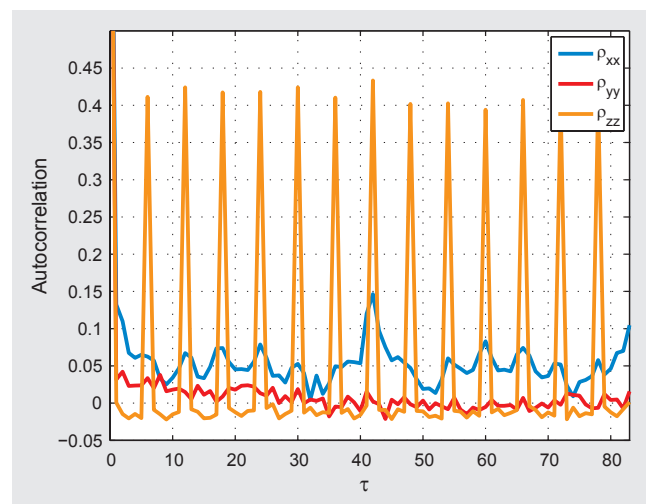


Fig. 3: $\hat{\rho}_{xx}, \hat{\rho}_{yy}$ and $\hat{\rho}_{zz}$ up to $\tau = 83$ for repetitions 1-330

Tab. 4: Maximum and number of absolute values of correlations

Correlation matrix of	max. correl. of coord. of grid points	max. correl. of remain. coord.	number of correl. < 0.02	number of correl. from 0.02 to 0.10
measurements	0.990	0.769	1 763	4 392
system. effects	0.303	0.424	4 815	2 726

viously, the same errors appear for all horizontal angles what results in peaks of correlations with time differences equal to the time between the scans of adjacent columns of the grid.

As mentioned above, the expected values $\hat{\mu}$ and the covariance matrix $\hat{\Sigma}$ in (34) of the measurements of the individual coordinates of the grid points are estimated by the repetitions 1-330, thus, $n_w = 330$ in (1) and (2). Information on the absolute values of the correlations following from the covariances are given in the first line of Tab. 4. The three coordinates of the grid points are highly correlated as pointed out in the introduction. Most of the remaining 7 749 correlations above the diagonal of the correlation matrix are small, see Tab. 4. The variances and covariances of the x -, y - and z -coordinates are obtained by the auto- and cross-covariances estimated by (14) and (16) from $n_s = n_g \times n_w = 13\,860$ measurements. The resulting correlation matrix \hat{R}_e in (19) is representative for the dependencies of the coordinates. It is therefore used as the correlation matrix for the systematic effects of the coordinates. This correlation matrix is smooth in comparison to the correlation matrix of the measurements. It contains smaller absolute values of correlations as shown in the second line of Tab. 4.

As mentioned in the introduction, the sum of the distances of a laser scanner to the measured points is very sensitive to correlations. The uncertainty and the expanded uncertainty of the sum of distances are therefore computed by Monte Carlo methods based on the measured coordinates of the $n_g = 42$ grid points of the repetitions 1-330. The results are expressed by standard deviations and coverage intervals which refer to the estimated expected values of the sum. They are computed for the measurements only and for the measurements plus systematic effects. First, independent measurements with univariate normal distributions with the expected values

$\hat{\mu}$ from (1) and the variances from the diagonal elements of the covariance matrix $\hat{\Sigma}$ in (3) are assumed. The results are given in the first line of Tab. 5. Correlated measurements with the multivariate normal distribution (34) with the expected values from (1) and covariance matrix from (3) are then applied. The results are shown in the second line of Tab. 5. They indicate that the standard deviation and the coverage limits considerably increase in comparison to the first line. To demonstrate that the results of the second line of Tab. 5 are governed by the variances and covariances of y because of the small variances for x and z , see Tab. 2, all covariances except the ones of the y -coordinates of different grid points are set equal to zero. The results are written in the third line of Tab. 5. They agree with the ones of the second line except for the lower limit of the coverage interval. This difference is caused by the randomness of the Monte Carlo methods.

Independent measurements with univariate normal distributions and independent systematic effects with standard deviations from (37) and with univariate rectangular distributions are then applied. The results are shown in the fourth line of Tab. 5. Random variates from the multivariate normal distribution (34) and the multivariate rectangular distribution (35) lead to the results of the last line of Tab. 5. The correlations estimated from the random variates for the systematic effects differ slightly from the prescribed correlations matrix \hat{R}_e from (19). The maximum absolute difference of the correlations is 0.022, while the square root of the mean squared differences is 0.004. The agreement is sufficient, because the prescribed correlations are estimated. The multivariate rectangular distribution (35) therefore possesses indeed the prescribed correlation matrix.

Introducing systematic effects increases the standard deviations by 3.3 mm in case of independency and by 4.0 mm for dependency. The difference of the standard

Tab. 5: Standard deviations and 0.95% coverage limits in mm for the sum of distances to the grid points

Distribution	Standard deviation	0.95% coverage interval	
		lower limit	upper limit
univariate normal	16.1	-31.6	31.8
multivariate normal	24.0	-47.5	47.1
multivariate normal for y	24.0	-46.9	47.1
univ. normal & univ. rectangular	19.4	-37.9	37.9
multiv. normal & multiv. rectangular	28.0	-54.5	54.8

deviations between the first and the last line of Tab. 5 is 11.9 mm. Assuming independent measurands when they are actually correlated gives uncertainties and expanded uncertainties which might be too optimistic.

As mentioned above, the expected values for the sum of the distances to the grid points are also computed by the five different distributions for Tab. 5. The results vary between 474 033.0 mm and 474 033.2 mm due to the randomness of the Monte Carlo method. Each result in Tab. 5 is computed by generating $m = 100\,000$ random variates. This number has been found to be sufficiently high to give the standard deviations and the coverage limits with at least two significant digits (Koch 20008b).

The repeated measurements of the multivariate linear model are assumed as being uncorrelated. To check whether this is fulfilled, the expected values and the covariance matrix of the measurements in (34) are estimated by (1) and (3) and the correlation matrix of the systematic effects by (19) with every second repetition of the first block of observations 1-330, thus by $n_w = 165$ repetitions. It introduces additional time between the repetitions to assure independency. The results thus obtained differ from the results of Tab. 5 by less than 0.9 mm for the standard deviations and less than 1.8 mm for the coverage limits. It indicates that each repetition is independent from the following one. This is confirmed by Fig. 3 and 4 which show small values of $\hat{\rho}_{yy}$ for $\tau = 42$ to $\tau = 83$ of the second repetition.

5 Conclusions

The sum of the distances of the laser scanner to the grid points is computed by the measured coordinates. The uncertainty of the sum is expressed by its standard deviation and the extended uncertainty by its coverage interval. The uncertainty of the sum increases if correlated measurements are introduced instead of independent measurements. The uncertainty increases again by considering in addition correlated systematic effects. To avoid uncertainties of the results of the measurements of laser scanners which are too optimistic, correlated measurements plus correlated systematic effects should be introduced. The results are obtained by the first block of repeated measurements, the correlations of which do not seem to be distorted by systematic effects. Nevertheless, systematic effects cannot be completely excluded. It would mean that the results for the standard deviations and the coverage intervals computed by the correlated measurements and the correlated measurements plus correlated systematic effects are too pessimistic.

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