# New Solutions to the Direct and Indirect Geodetic Problems on the Ellipsoid

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**Summary**
Taking advantage of numerical integration solves the direct and indirect geodetic problems on the ellipsoid. In general the solutions are composed of a strict solution for the sphere plus a correction to the ellipsoid determined by numerical integration. The solutions are demonstrated by three numerical examples.

**Zusammenfassung**
*Die klassische Berechnung der geodätischen Hauptaufgaben auf dem Rotationsellipsoid beruht auf der Entwicklung der Integranden der elliptischen Integrale in Taylor-Reihen. In diesem Beitrag wird die exakte Lösung für die Kugel mit Korrekturen für das Ellipsoid, die durch numerische Integration bestimmt werden, kombiniert. Als eine weitere Anwendung kann die Fläche eines geodätischen Polygons mit dieser Methode berechnet werden. Die Lösungen werden anhand von drei numerischen Beispielen veranschaulicht.*

## 1 Introduction

The direct and indirect geodetic problems on the ellipsoid have attracted the attention of numerous geodesists in the past. Although ellipsoidal solutions are of less importance in the space age, problems related with geodesics may still be relevant. Examples can be found e. g. in the application of the Law of the Sea, military surveys and in the optimisation of aircrafts' and ships' routes.

The classical solutions of these problems were frequently in the forms of power series of the ellipsoidal eccentricity. Some of the basic formulas used below are provided e. g. by Heck (1987). Today it is natural to take advantage of numerical integration by computers. Klotz (1991) and Schmidt (1999) presented some pioneer work along this line, and our solutions to the direct and indirect problems will also rely on numerical integrations. Similarly, the area under the geodesic can be conveniently computed by numerical integration (Sect. 4).

## 2 The direct problem on the ellipsoid

Let us assume that the ellipsoidal parameters a and b, the semi-major and -minor axes, are given. From these we compute the first eccentricity by $e=\sqrt{a^2-b^2}/a$. More details on the basic formulas used are presented in Heck (1987), Klotz (1991) and Schmidt (1999).

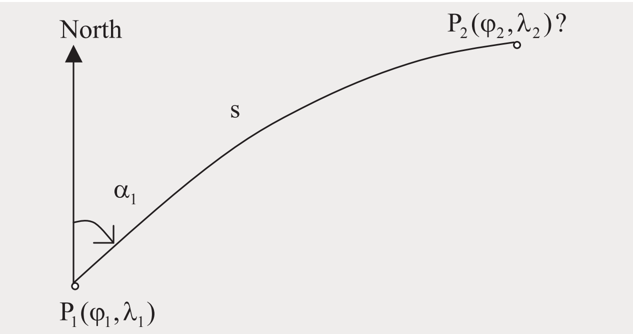


**Fig. 1: The direct problem on the ellipsoid**

**Problem:** Let us assume that the coordinates of latitude $(\varphi_1)$ and longitude $(\lambda_1)$ of point $P_1$ are given. Also given are the azimuth $\alpha_1$ at $P_1$ and the geodesic arc length s to another point $P_2$. Determine the coordinates of $P_2$ (see Fig. 1).

**Solution:**
a) Let us first convert some trigonometric functions of the geodetic latitude $\varphi$ to functions of the reduced latitude $\beta$:

$$\tan\beta=k\tan\varphi\,,$$

$$\sin\beta=k\sin\varphi/\sqrt{1-e^2\sin^2\varphi}\quad\text{and}$$

$$\cos\beta=\cos\varphi/\sqrt{1-e^2\sin^2\varphi} \tag{1}$$

where $k=\sqrt{1-e^2}$.

b) Compute Clairaut's constant for the geodesic by the formula

$$c=\cos\beta_1\sin\alpha_1\,. \tag{2}$$

If $\beta_1\neq\beta_2$ and starting from the following equation for the given arc length s (Heck 1987 and Schmidt 1999):

$$s=a\int_{\beta_1}^{\beta_2}\sqrt{\frac{1-e^2\cos^2\beta}{\cos^2\beta-c^2}}\cos\beta d\beta\,, \tag{3a}$$

we arrive at the formula

$$s=a\left[\arcsin\frac{x}{\sqrt{1-c^2}}\right]_{x=x_1}^{x=x_2}+ds\;, \tag{3b}$$

where

$$ds=-ae^2\int_{x_1}^{x_2}\frac{(1-x^2)dx}{\sqrt{1-c^2-x^2}\left(1+\sqrt{1-e^2+e^2x^2}\right)}\,. \tag{3c}$$

Hence, $\beta_2$ can be determined by

$$\sin\beta_2=\sqrt{1-c^2}\sin\left[(s-ds)/a+\arcsin\left(x_1/\sqrt{1-c^2}\right)\right]\;, \tag{4}$$

where ds is preferably determined by numerical integration. As the upper integration limit depends on $\beta_2$, Eq. (4) should be iterated. As a starting value ds can be set to 0. [One problem of Eq. (3c) is the singularity of the integral for $x=\sqrt{1-c^2}$.]

If $\beta_1=\beta_2$, of course, the above procedure does not hold, but $\beta_1=\beta_2$ can be checked by the formula

$$s=a\left(\pi-2\arcsin\frac{x_1}{\sqrt{1-c^2}}\right)+ds_1\;, \tag{5a}$$

where

$$ds_1=-2ae^2\int_{x_1}^{\sqrt{1-c^2}}\frac{(1-x^2)dx}{\sqrt{1-c^2-x^2}\left(1+\sqrt{1-e^2+e^2x^2}\right)}\,. \tag{5c}$$

<u>Remark:</u> If $\beta_1\approx\beta_2$ the iteration of Eqs. (3b) and (3c) may be very slow or fail, which calls for a special investigation out of the scope of this article.

c) If $\beta_1\neq\beta_2$, the longitude $\lambda_2$ is given by the formula (e.g. Heck 1987 or Schmidt 1999)

$$\begin{aligned}\lambda_2&=\lambda_1+c\int_{\beta_1}^{\beta_2}\sqrt{\frac{1-e^2\cos^2\beta}{\cos^2\beta-c^2}}\frac{d\beta}{\cos\beta}\\&=\lambda_1+c\int_{\beta_1}^{\beta_2}\frac{d\beta}{\cos\beta\sqrt{\cos^2\beta-c^2}}+d\lambda\\&=\lambda_1+\left[\arcsin\frac{t_i}{t_0}\right]_{i=1}^{i=2}+d\lambda\end{aligned} \tag{6a}$$

where

$$d\lambda=-e^2c\int_{x_1=\sin\beta_1}^{x_2=\sin\beta_2}\frac{dx}{\sqrt{1-c^2-x^2}\left(1+\sqrt{1-e^2+e^2x^2}\right)}\,, \tag{6b}$$

$$t_i=\tan\beta_i\quad\text{and}\quad t_0=\sqrt{1-c^2}/c\,.$$

The integral of Eq. (6b) can be determined by numerical integration between the limits $x_1$ and $x_2$.

If $\beta_1=\beta_2\neq0$, the longitude of the new point is given by

$$\begin{aligned}\lambda_2&=\lambda_1+2c\int_{x_1}^{\sqrt{1-c^2}}\frac{dx}{(1-x^2)\sqrt{1-c^2-x^2}}+d\lambda'\\&=\lambda_1+2\left[\frac{\pi}{2}-\arcsin\frac{t_1}{t_0}\right]+d\lambda'\end{aligned} \tag{7a}$$

where

$$d\lambda' = -2e^2c \int_{x_1}^{\sqrt{1-c^2}} \frac{dx}{\left(\sqrt{1-c^2-x^2}\right)\left(1+\sqrt{1-e^2+e^2x^2}\right)} . \quad (7b)$$

As an alternative to the above equations we may take advantage of the similarities of Eqs. (3a) and (6a) to derive

$$\Delta\lambda = \frac{c}{a}s + I + d\lambda_1 \ , \quad (8a)$$

where

$$I = \begin{cases} c\int_{x_1}^{x_2} \frac{x^2dx}{(1-x^2)\sqrt{1-c^2-x^2}} = \left(\arcsin\frac{t_i}{t_0} - c\arcsin\frac{x_i}{\sqrt{1-c^2}}\right)\Big|_{i=1}^{i=2} , \\ x_1 \neq x_2 \\ 2c\int_{x_1}^{\sqrt{1-c^2}} \frac{x^2dx}{(1-x^2)\sqrt{1-c^2-x^2}} = \pi(1-c) + 2\left(c\arcsin\frac{x_1}{\sqrt{1-c^2}} - \arcsin\frac{t_1}{t_0}\right), \\ x_1 = x_2 . \end{cases} \quad (8b)$$

and

$$d\lambda_1 = \begin{cases} -ce^2\int_{x_1}^{x_2} \frac{x^2dx}{\sqrt{1-c^2-x^2}\left(1+\sqrt{1-e^2+e^2x^2}\right)} & \text{if} \quad x_1 \neq x_2 \\ -2ce^2\int_{x_1}^{\sqrt{1-c^2}} \frac{x^2dx}{\sqrt{1-c^2-x^2}\left(1+\sqrt{1-e^2+e^2x^2}\right)} & \text{if} \quad x_1 = x_2 . \end{cases} \quad (8c)$$

In the special case that both given points are located on the equator Eq. (8a) reduces to

$$\Delta\lambda = \frac{cs}{a} + \pi(1-c) - 2ce^2 \int_0^{\sqrt{1-c^2}} \frac{x^2dx}{\sqrt{1-c^2-x^2}\left(1+\sqrt{1-e^2+e^2x^2}\right)} , \quad (9)$$

or, approximately,

$$\Delta\lambda \approx \frac{cs}{a} + \pi(1-c) - ce^2 \int_0^{\sqrt{1-c^2}} \frac{x^2dx}{\sqrt{1-c^2-x^2}} = \frac{cs}{a} + \pi(1-c) - \frac{c(1-c^2)\pi}{4}e^2 . \quad (10)$$

This completes the solution of the direct problem on the ellipsoid.

## 3 The indirect problem

In the indirect problem the coordinates of two points (say $P_1$ and $P_2$) are given.

**Problem:** Determine the arc length along the geodesic between the two points and the azimuths of the geodesic at the two points.

**Solution:**

a) First convert the geodetic latitudes $\varphi_i$ to reduced latitudes $\beta_i$.

b) Then determine Clairaut's constant for the geodesic running through the two given points (Sjöberg 2005b). Three cases must be distinguished.

i) In the general case with $\beta_1 \neq \beta_2$, compute

$$t_0^2 = \frac{t_1^2 + t_2^2 - 2t_1^2t_2^2\cos DL}{\sin^2 DL} , \quad (11a)$$

where $t_i = \tan\beta_i$,

$$DL = \lambda_2 - \lambda_1 - d\lambda_c \quad (11b)$$

and

$$d\lambda_c = -e^2c\int_{x_1}^{x_2} \frac{dx}{\sqrt{(1-c^2-x^2)}\left(1+\sqrt{1-e^2+e^2x^2}\right)} . \quad (11c)$$

Eq. (11c), with $x_i = \sin\beta_i$, is conveniently integrated numerically. Clairaut's constant is then given by

$$c = 1/\sqrt{1+t_0^2} \ . \quad (12)$$

ii) If $\beta_1 = \beta_2 \neq 0$, Clairaut's constant is given by the equation

$$c = \frac{\cos(D\lambda)}{\sqrt{t_1^2 + \cos^2(D\lambda)}} , \quad (13a)$$

where (with $x_1 = \sin\beta_1$)

$$D\lambda = \frac{\lambda_2 - \lambda_1}{2} + e^2c\int_{x_1}^{\sqrt{1-c^2}} \frac{dx}{\sqrt{(1-c^2-x^2)\left(1+\sqrt{1-e^2+e^2x^2}\right)}} . \quad (13b)$$

iii) If $\beta_1 = \beta_2 = 0$, Clairaut's constant is given by

$$c = \frac{\cos(DL)}{\sqrt{t_{11}^2 + \cos^2(DL)}} , \quad (14a)$$

where $t_{11} = \tan d\beta_{11}$, $d\beta_{11} < \beta_{max}$ being an arbitrary but small latitude, and

$$DL = \frac{\lambda_2 - \lambda_1}{2} + e^2c\int_0^{\sqrt{1-c^2}} \frac{dx}{\sqrt{(1-c^2-x^2)(1-e^2+e^2x^2)}} - \frac{cd\beta_{11}}{\sqrt{1-c^2}} . \quad (14b)$$

In all three cases the procedures require iteration to reach the final determination of c.

c) The arc length follows from Eqs. (3a) and (3b). If $\beta_1 = \beta_2$, we may take advantage of Eqs. (8a)-(8c) to derive

$$s = \frac{a}{c}(\Delta\lambda - I - d\lambda_1). \tag{15}$$

In particular, if $\beta_1 = \beta_2 = 0$, the arc length can be approximated by

$$s \approx \frac{a}{c}\left[\Delta\lambda - \pi(1-c) + (2-c)(1-c^2)\frac{\pi e^2}{4}\right]. \tag{16}$$

d) The azimuths at points $P_1$ and $P_2$ are given by Clairaut's equation, Eq. (2):

$$\sin\alpha_i = c/\cos\beta_i . \tag{17}$$

This completes the solution to the indirect geodetic problem on the ellipsoid.

## 4 Area computation under the geodesic

Following Sjöberg (2005a) the area (T) on the ellipsoid limited by the equator, a geodesic with the Clairaut constant c and the two meridians with longitudes $\lambda_1$ and $\lambda_2 > \lambda_1$ can be written

$$T = b^2\int_{\lambda_1}^{\lambda_2} f(\varphi)d\lambda = b^2(\alpha_2 - \alpha_1) + b^2\int_{\lambda_1}^{\lambda_2}[f(\varphi) - \sin\varphi]d\lambda , \tag{18a}$$

where

$$f(\varphi) = \frac{\sin\varphi}{2(1-e^2\sin^2\varphi)} + \frac{1}{4e}\ln\frac{1+e\sin\varphi}{1-e\sin\varphi} . \tag{18b}$$

In the last equality of Eq. (18a) we have taken advantage of the following differential property of the geodesic:

$$\sin\varphi d\lambda = d\alpha . \tag{19}$$

Expressing $d\lambda$ in terms of latitude [e.g. Sjöberg 2005a, Eq. (19)], we finally arrive at the solution

$$T = b^2(\alpha_2 - \alpha_1) + b^2ck\int_{\varphi_1}^{\varphi_2}\frac{f(\varphi) - \sin\varphi}{g(x)}d\varphi , \tag{20a}$$

where

$$g(x) = x\sqrt{\left[1-e^2(1-x^2)\right]\left[(1-c^2) - (1-e^2c^2)(1-x^2)\right]} , \tag{20b}$$

$$k = \sqrt{1-e^2} \text{ and } x = \cos\varphi .$$

Alternatively, Eq. (20a) can be written

$$T = b^2(\alpha_2 - \alpha_1) + b^2ck\int_{x_2}^{x_1}\frac{f_1(x) - 1}{g(x)}dx , \tag{21a}$$

where

$$f_1(x) = \frac{1}{2(1-e^2+e^2x^2)} + \frac{1}{4e\sqrt{1-x^2}}\ln\frac{1+e\sqrt{1-x^2}}{1-e\sqrt{1-x^2}} . \tag{21b}$$

In the above equations $b = a\sqrt{1-e^2}$, $\varphi_1$ and $\varphi_2 > \varphi_1$ are the latitudes at the intersections of the meridians and the geodesic, $x_i = \cos\varphi_i$, $\alpha_i = \arcsin(c/\cos\beta_i)$ and

$$\cos\beta_i = x_i/\sqrt{1-e^2(1-x_i^2)} . \tag{22}$$

The integral terms of Eqs. (20a) and (21a), of order $e^2$, are suitable for numerical integration. Note that $f_1(x)$ is well conditioned for any x in the interval $0 \le x \le 1$ with $f_1(1) = 1$.

If $\varphi_1 = \varphi_2$, Eqs. (20a, b) and (21a, b) are modified to

$$T = b^2(\pi - 2\alpha_1) + 2b^2ck\int_0^{\varphi_{max}}\frac{f(\varphi) - \sin\varphi}{g(\varphi)}d\varphi \tag{23}$$

and

$$T = b^2(\pi - 2\alpha_1) + 2b^2ck\int_{\cos\varphi_{max}}^{x_1}\frac{f_1(x) - 1}{g(x)}dx , \tag{24}$$

where

$$\cos\varphi_{max} = kc/\sqrt{1-e^2c^2} \text{ and } c = \sin\alpha_1 . \tag{25}$$

## 5 Numerical examples

We will present three numerical examples on the direct and indirect problems and area computation on the ellipsoid with a = 6378 km and $e^2$ = 0.00694.

**Example 5.1:**
a) Let $P_1$ be defined by $\beta_1 = \lambda_1 = 0$, s = 1000 km and $\alpha_1 = 60°$. Compute $P_2$ and $\alpha_2$.

**Solution:** From Eq. (2) we get Clairaut's constant:
$c = \cos\beta_1\sin\alpha_1 = 0.866025$.
By iterating Eq. (3b) and (4) we obtain $\beta_2^{(0)} = 0.078153$, $\beta_2^{(2)} = \beta_2^{(3)} = \beta_2 = 0.078524$, which yields $\beta_2 = 4°29'56''.81$.
Then the longitude of $P_2$ is given by Eqs. (6a) and (6b), yielding $\lambda_2 = 0.136239 = 7°48'21''.45$.

b) Let $P_1$ and $P_2$ be defined from a). Compute s and $\alpha_1$.

**Solution:** First c is iterated twice by Eqs. (11) and (12) to 0.866025. This yields
$\alpha_1 = \arcsin(c/\cos\beta_1) = \arcsin 0.866025 = 59°59'59''.83$.
Then the arc length is determined by Eqs. (3a) and (3b) to s = 999.9999979 km.

**Example 5.2:**
a) Let $\lambda_1 = \beta_1 = \beta_2 = 0$, $\lambda_2 = 50°$. Compute s and $\alpha_1$.

**Solution:** First c is determined by Eqs. (14a) and (14b). As the integrand of Eq. (14b) becomes singular at the upper integration limit, it was approximated by $\sqrt{1-c^2} - 0.005$. After 2 iterations c = 0.999403 had stabilized. Then the arc length and the azimuth at $P_1$ were computed by Eqs. (16) and (2) to s = 5557.26246 km and $\alpha_1 = 1.536261 = 88°1'16''.62$, respectively.

b) Let $P_1$, s and $\alpha_1$ be given according to a). Compute the coordinates of $P_2$.

**Solution:** First the latitude of $P_2$ was searched by Eq. (4), which did not converge. Then, by *assuming* that also $\beta_2 = 0$, Eq. (10) yielded $\lambda_2 = 50°.00'00''.00$. Then, using Eq. (16), we obtained s = 5557.26247, which value thus agrees with its given value in a). Thus we have verified that $P_2$ is located on the equator, and the direct problem is solved.

**Example 5.3:**
Compute the area under the geodesic limited by the meridians through $P_1$ and $P_2$ in *Example 5.1*.

**Solution:** The azimuths are given by
$\alpha_1 = \arcsin(c/\cos\beta_1) = 1.04719674$ and
$\alpha_2 = \arcsin(c/\cos\beta_2) = 1.05257549$.
From $b = a\sqrt{1-e^2}$ one obtains the approximate area
$T_1 = b^2(\alpha_2 - \alpha_1) = 217283.20$ km².

Furthermore, $\varphi_1 = 0$ and
$\varphi_2 = \arctan\left(\tan\beta_2/\sqrt{1-e^2}\right) = 0.0787968$, which inserted into Eqs. (20a) and (20b) yields

$$T_2 = bck\int_{\varphi_1}^{\varphi_2} \frac{f(\varphi) - \sin\varphi}{g(\varphi)} d\varphi = 2.36 \text{ km}^2$$

and, finally,

$$T = T_1 + T_2 = 217285.56 \text{ km}^2.$$

## 6 Concluding remarks

We have demonstrated how to solve the direct and indirect problems on the ellipsoid by adding the strict solution for the sphere and an ellipsoidal correction determined by numerical integration. By employing numerical integration, the routines of which are usually available in current computer software like MATLAB®, the problems of classical geodesy are easily solved to desired precision. Remaining problems to be solved are related with the case that the two points of interest on the geodesic are located at nearly the same latitude, implying that the proposed solutions for the direct and indirect problems become numerically unstable.

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