Closed form solution of the triple three-dimensional intersection problem

Joseph L. Awange, Erik W. Grafarend and Yoichi Fukuda

Summary

The *reduced Groebner basis* is here applied to offer a closed form solution to the three-dimensional intersection problem vital in Photogrammetry and Computer vision. From the observations of type *horizontal directions* T_i and *vertical directions* B_i, with $i = 1, 2, 3$, we demonstrate that the three *nonlinear system of equations* can be decomposed to three *quartic polynomials*, i. e. polynomials of degree four, whose roots can be obtained with the help of *solve* command in *Matlab* software. The advantage is that when one is faced with the minimum number of known points, here three, one can still carry out an intersection to obtain the coordinates of the desired point. In Photogrammetry, the procedure could be used to obtain the coordinates of pass points where known stations are limited only to the minimum number.

Zusammenfassung

Im vorliegenden Beitrag wird eine reduzierte Gröbner Basis benutzt, um eine geschlossene Lösung des für Photogrammetrie und Computervision zentralen Problems des dreidimensionalen Geradenschnittproblems herzuleiten. Es wird gezeigt, dass die drei nichtlinearen Gleichungen, die die beobachteten Horizontalrichtungen T_i *und Vertikalrichtungen* B_i, $i = 1, 2, 3$ *mit der gesuchten Position verknüpfen, in drei Polynome vierten Grades zerlegt werden können. Die Nullstellen dieser Polynome können durch das »solve«-Kommando der MATLAB-Software bestimmt werden. Der Vorteil der beschriebenen Vorgehensweise besteht darin, dass bereits beim Vorliegen der minimalen Anzahl von drei bekannten Punkten das Geradenschnittproblem gelöst werden kann. Dieser Algorithmus kann in der Photogrammetrie dazu genutzt werden, die Koordinaten von Passpunkten zu bestimmen, wenn nur drei Stationen mit bekannten Koordinaten zur Verfügung stehen.*

1 Introduction

Thanks to the *Global Positioning System* (GPS) classical geodetic and photogrammetric positioning techniques have reached a new horizon. Geodetic and photogrammetric direction observations (Machine Vision, »Total Observing Stations«) have to be analyzed in a *three-dimensional Euclidean Space*. The pair of tools called *»Resection and Intersection«* has to operate three-dimensionally. Closed form solutions of the *three-dimensional resection problem* exist in a great number (Awange 2002a,b and Awange and Grafarend 2003 (in press a,b)), but closed form solutions of the *three-dimensional intersection problem* are very rare. For instance, Grafarend and Shan (1997) solved the two $P4P$ or the combined three-dimensional resection-intersection problem in terms of *Moebius barycentric coordinates* in a closed form. One reason for the rare existence of the closed form solution of the three-dimensional intersection problem is the nonlinearity of the directional observational equations, partially caused by the external *orientation parameters*. One target of our contribution is accordingly to address the problem of orientation parameters.

The key to overcome the problem of nonlinearity caused by *orientation parameters* is taken from the *Baarda Doctrine*. W. Baarda (1967, 1973) proposed to use *dimensionless quantities* in geodetic and photogrammetry networks: Angles in a three-dimensional *Weitzenboeck space*, shortly called *space angles*, as well as *distance ratios* are the *dimensionless* structure elements which are *equivalent* under the action of the seven parameter *conformal group*, also called similarity transformation.

For the *two-dimensional intersection problem* (Awange (2003)), the closed form solution in terms of angles has a long tradition. Consult Fig. 1 where we introduce the angles ψ_{12} and ψ_{21} in the planar triangle $\Delta : P_0P_1P_2$. P_0, P_1, P_2 are the nodes: The Cartesian coordinates (x_1, y_1) and (x_2, y_2) of the points P_1 and P_2 are given, the Cartesian coordinates (x_0, y_0) of the point P_0 are unknown. The angles $\psi_{12} = \alpha$ and $\psi_{21} = \beta$ are derived from direction observations by differencing horizontal. $\psi_{12} = T_{10} - T_{12}$ or $\psi_{21} = T_{20} - T_{21}$ are examples of observed horizontal directions T_{10} and T_{12} from P_1 to P_0 *and* P_1 to P_2 or T_{21} to T_{20} from P_2 to P_1 and P_2 to P_0. *By means of taking differences we map direction observations to angles and eliminate orientation unknowns.* The solution of the two-dimensional intersection problem *in terms of angles*, a classic in analytical surveying, is given by equation (1-1) and (1-2) as

$$x_0 = s_{12}\frac{cos\alpha sin\beta}{sin(\alpha+\beta)} \quad (1\text{-}1)$$

$$y_0 = s_{12}\frac{sin\alpha sin\beta}{sin(\alpha+\beta)}. \quad (1\text{-}2)$$

Note the Euclidean distance between the nodal points, namely $s_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ with the origin being at P_1 and the X-axis pointing to the direction P_2.

For the *three-dimensional intersection problem*, the problem of transferring observed horizontal *and* vertical directions to space angles or of images of coordinates in a photogram to space angles has already been solved (Awange

2002a, Grafarend and Shan 1997, Awange and Grafarend 2003 and Awange and Grafarend (in press a,b)). Equations (1-3) and (1-4) below are the analytical versions of the map of directions or image coordinates to space coordinates. Indeed, the *map eliminates the external orientation parameters*. The *space angle in terms of horizontal and vertical directions* is given by

$$\cos\psi_{12} = \cos B_1 \cos B_2 \cos(T_2 - T_1) + \sin B_1 \sin B_1 \tag{1-3}$$

while the *space angle in terms of image coordinates/perspective coordinates* $(x_1, y_1), (x_2, y_2)$ *and the focal length* f" is given by

$$\cos\psi_{12} = \frac{x_1x_2 + y_1y_2 + f^2}{\sqrt{x_1^2 + y_1^2 + f^2}\sqrt{x_2^2 + y_2^2 + f^2}}. \tag{1-4}$$

Here, we present you with a closed form solution of the *three-dimensional intersection* problem where a triple of three points P_1, P_2, P_3 are given by their three-dimensional Cartesian coordinates $X_1, Y_1, Z_1, X_2, Y_2, Z_2, X_3, Y_3, Z_3$, but the coordinates of the zero point X_0, Y_0, Z_0 are unknown. The *dimensionless quantities* $\psi_{12}, \psi_{23}, \psi_{31}$ are space angles $\psi_{12} = \angle P_0P_1P_2$, $\psi_{23} = \angle P_0P_2P_3$, $\psi_{31} = \angle P_1P_3P_0$ which are derived from the measurements as outlined above.

Section 2 outlines the quadratic observational equations for space angles which are solved for the distances P_0P_1, P_0P_2, P_0P_3 by means of *reduced Groebner basis* approach outlined in Section 3 (Awange 2002a,b). Section 4 presents the solution of the three-dimensional intersection problem where the distances are determined by *Groebner basis algorithm* and the station coordinates by the three-dimensional ranging *Awange-Grafarend algorithm*. Both algorithms give closed form solutions. In Section 5, an example is presented and the study concluded in Section 6.

Our contribution of solving the *three-dimensional intersection problems* extends the earlier results of Grafarend (1990), Grafarend and Mader (1993) and Grafarend and Shan (1997).

2 Three-dimensional 3-point intersection

This problem is formulated as follows: Given three *space angles* $\{\psi_{12}, \psi_{23}, \psi_{31}\}$ obtained from the spherical coordinates of type horizontal directions T_i and vertical directions B_i, for $i = 1, \ldots, 3$ in Fig. 1, obtain the distances $\{x_1 = S_1, x_2 = S_2, x_3 = S_3\}$ from the unknown point $P \in \mathbb{E}^3$ to three other known stations $P_i \in \mathbb{E}^3 \mid i = 1, 2, 3$ in the first step. In the second step, the obtained distances from step 1 are treated as pseudo-observations. From an unknown point $P \in \mathbb{E}^3$ to a minimum of three known points $P_i \in \mathbb{E}^3 \mid i = 1, 2, 3$ in Fig. 1, determine the position $\{X, Y, Z\}$ of the unknown point $P \in \mathbb{E}^3$. When only three known stations are used to determine the position of

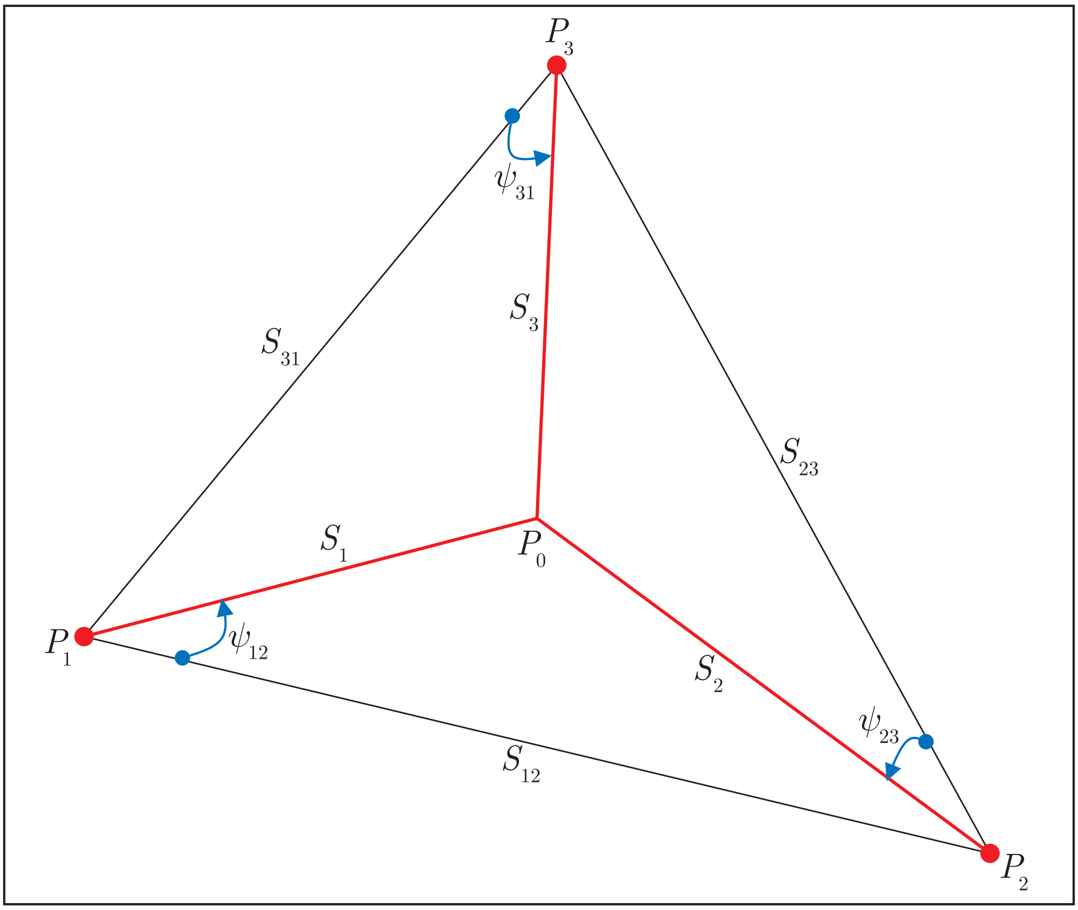

Fig. 1: 3-point intersection

the unknown station in three-dimension, the problem reduces to that of 3d closed form solution. First, we develop the algebraic form of the problem by converting the nonlinear system of equations into *polynomial*. We point out here that the unknowns $\{X, Y, Z\}$ could also be written directly in terms of the space angles but this is still subject to study and will be presented in coming works.

From Fig. 1, the nonlinear system of equations for the three-dimensional 3-point positioning is given as

$$
\begin{aligned}
x_2^2 &= x_1^2 + S_{12}^2 - 2S_{12}\cos(\psi_{12})x_1 \\
x_3^2 &= x_2^2 + S_{23}^2 - 2S_{23}\cos(\psi_{23})x_2 \\
x_1^2 &= x_3^2 + S_{31}^2 - 2S_{31}\cos(\psi_{31})x_3
\end{aligned}
\qquad (2\text{-}1)
$$

which can be expressed in algebraic form without bilinear terms as

$$
\begin{aligned}
x_1^2 - 2S_{12}\cos(\psi_{12})x_1 - x_2^2 + S_{12}^2 &= 0 \\
x_2^2 - 2S_{23}\cos(\psi_{23})x_2 - x_3^2 + S_{23}^2 &= 0 \\
x_3^2 - 2S_{31}\cos(\psi_{31})x_3 - x_1^2 + S_{31}^2 &= 0,
\end{aligned}
\qquad (2\text{-}2)
$$

$\{x_1 = S_1, x_2 = S_2, x_3 = S_3\}$ being the distances from P_1 to P_0, P_2 to P_0 and P_3 to P_0 respectively. Equation (2-2) can be solved by adding (2-2)i, (2-2)ii and (2-2)iii to eliminate the squared terms to get a linear equation in x_1, x_2 and x_3. x_1 is then expressed in terms of x_2 and x_3 and substituted in (2-2)i to give an expression in x_2 and x_3 only. The resulting expression in x_2 and x_3 can then be solved simultaneously with (2-2)ii to give x_2 and x_3. On the other hand, if the linear equation in x_1, x_2 and x_3 is now written such that x_3 is then expressed in terms of x_2 and x_1 and substituted in (2-2)iii, an expression in x_2 and x_1 will be given which together with (2-2)i can be solved for x_2 and x_1.

The setback with this approach is that one parameter, in this case x_2, will be determined twice with differing values being given. This clearly is undesirable and one would wish to have a direct method of solving the problem. We present in the next section the polynomial approach of *Groebner basis* that offers a direct solution to the problem.

3 Groebner basis

As a recipe to what *Groebner bases* can do, consider that most problems in nature, here in Geodesy, Photogrammetry, Machine Vision, Robotics, Surveying can be modelled by a *set of nonlinear equations* forming polynomials. This *nonlinear system of equations* that have to be solved can be used to form a linear combination of other polynomials called *Ideals* by being multiplied by *arbitrary polynomials* and summed up. In this case, a collection of these *nonlinear algebraic equations* forming *Ideals* are referred to as the set of polynomials generating the *Ideal* and forms the elements of this *Ideal*. The *B. Buchberger algorithm* then takes this set of generating polynomials and derive another set of polynomials called *Groebner basis* which has some special properties. One of the special properties of the *Groebner bases* is that its elements can divide the elements of the generating set giving zero remainder. This property is achieved by the *B. Buchberger algorithm* by cancelling the *leading terms* of the polynomials in the generating set and in doing so deriving the *Groebner basis* of the *Ideal* (whose elements are the generating *nonlinear algebraic equations*). With the *lexicographic* type of ordering chosen, one of the elements of the *Groebner basis* is often a *univariate polynomial* which can be solved for the unknowns. The other special property is that two sets of polynomial equations will generate the same *Ideal* if and only if their *Groebner bases* are equal with respect to any term ordering. This property is important in that the solution of the *Groebner basis* will satisfy the original solution required by the generating set of nonlinear equations.

The *B. Buchberger algorithm*, more or less a generalization of the *Gauss elimination* procedure, makes use of the subtraction polynomials known as the *S-polynomials* to eliminate the leading terms of a pair of polynomials. In doing so and if *lexicographic ordering* is chosen, the process ends up with one of the computed *S-polynomials* being a univariate polynomial which can be solved and substituted back in the other *S-polynomials* using the *Extension Theorem* (Cox et al. 1998, pp. 25–26) to obtain the other variables.

In[1]:=GroebnerBasis[{polynomials}, {variables}, {options}] (where In[1]:= is the mathematica prompt) which computes the *Groebner basis* for the *Ideal* generated by the *polynomials* with respect to the *monomial order* specified by *monomial order options* with the *variables* specified as in the executable command gives the reduced *Groebner basis*. Without specifying the options part, one gets too many elements of the *Groebner basis* which may not be relevant. In Maple Version 5 the command is accessed by typing > *with (grobner);* (where > is the Maple prompt and the semicolon ends the Maple command). Once the *Groebner basis* package has been loaded, the execution command then becomes > *gbasis (polynomials, variables, termorder)* which computes the *Groebner basis* for the *Ideal* generated by the *polynomials* with respect to the *monomial ordering* specified by *termorder* and *variables* in the executable command. Following suggestions from *B. Buchberger* (1999), the Mathematica software is adopted in the present study. In order to compute the *Groebner basis* faster, one would opt for the *reduced Groebner basis*.

The *Groebner bases* approach adds to the treasures of methods that are used to solve *nonlinear algebraic systems of equations* in Geodesy, Photogrammetry, Machine Vision, Robotics and Surveying. For detailed literature on Groebner basis, we refer to standard text books on *Groebner bases* such as Davenport et al. (1998), Becker and Weispfenning (1998) and Cox et al. (1997, 1998).

4 Solution of the triple 3d-intersection problem

In order to obtain a solution to the problem, the polynomial approach of *reduced Groebner basis* proceeds in two steps.

Step 1:
The distances in (2-2) are solved by the use of *reduced Groebner basis* techniques discussed in Section 3 to obtain directly the distances x_1, x_2 and x_3. Equation (2-2) is rewritten as

$$\begin{aligned} f_1 &:= x_1^2 + b_1 x_1 - x_2^2 + a_0 = 0 \\ f_2 &:= x_2^2 + b_2 x_2 - x_3^2 + b_0 = 0 \\ f_3 &:= x_3^2 + b_3 x_3 - x_1^2 + c_0 = 0 \end{aligned} \qquad (4\text{-}1)$$

with $b_1 = -2S_{12}\cos(\psi_{12}), b_2 = -2S_{23}\cos(\psi_{23}), b_3 = -2S_{31}\cos(\psi_{31})$ and $a_0 = S_{12}^2, b_0 = S_{23}^2, c_0 = S_{31}^2$. Following Awange (2002b), the *reduced Groebner basis* is obtained from (4-1) by

$$\begin{aligned} &GroebnerBasis[\{f_1, f_2, f_3\}, \{x_1, x_2, x_3\}, \{x_1\}] \\ &GroebnerBasis[\{f_1, f_2, f_3\}, \{x_1, x_2, x_3\}, \{x_2\}] \\ &GroebnerBasis[\{f_1, f_2, f_3\}, \{x_1, x_2, x_3\}, \{x_3\}] \end{aligned} \qquad (4\text{-}2)$$

which leads to quartic equations for solving for the unknown distances given as

$$\begin{aligned} x_1 &:= d_4 x_1^4 + d_3 x_1^3 + d_2 x_1^2 + d_1 x_1 + d_0 = 0 \\ x_2 &:= e_4 x_2^4 + e_3 x_2^3 + e_2 x_2^2 + e_1 x_2 + e_0 = 0 \\ x_3 &:= f_4 x_3^4 + f_3 x_3^3 + f_2 x_3^2 + f_1 x_3 + f_0 = 0 \end{aligned} \qquad (4\text{-}3)$$

where the coefficients of (4-3) are as given in the Appendix. The results of this step are the distances $\{x_1 = S_i, x_2 = S_2, x_3 = S_3\}$ from the unknown point $P \in \mathbb{E}^3$ to three other known stations $P_i \in \mathbb{E}^3 \mid i = 1,2,3$ (Fig. 1) that we use in the next step to determine the position.

Step 2:
This step involves the ranging problem already discussed by Awange et al. (2003). Starting from three nonlinear 3d Pythagorus distance observation equations

$$\begin{aligned} S_1^2 &= (X_1 - X)^2 + (Y_1 - Y)^2 + (Z_1 - Z)^2 \\ S_2^2 &= (X_2 - X)^2 + (Y_2 - Y)^2 + (Z_2 - Z)^2 \\ S_3^2 &= (X_3 - X)^2 + (Y_3 - Y)^2 + (Z_3 - Z)^2 \end{aligned} \qquad (4\text{-}4)$$

relating the observed distances from step 1 to the three unknowns $\{X, Y, Z\}$, two equations with three unknowns are derived. Equation (4-4) is expanded in the form

$$\begin{aligned} S_1^2 &= X_1^2 + Y_1^2 + Z_1^2 + X^2 + Y^2 + Z^2 \\ &\quad -2X_1X - 2Y_1Y - 2Z_1Z \\ S_2^2 &= X_2^2 + Y_2^2 + Z_2^2 + X^2 + Y^2 + Z^2 \\ &\quad -2X_2X - 2Y_2Y - 2Z_2Z \\ S_3^2 &= X_3^2 + Y_3^2 + Z_3^2 + X^2 + Y^2 + Z^2 \\ &\quad -2X_3X - 2Y_3Y - 2Z_3Z \end{aligned} \qquad (4\text{-}5)$$

and differenced in (4-6) to eliminate the quadratic terms $\{X^2, Y^2, Z^2\}$.

$$\begin{aligned} S_1^2 - S_2^2 &= X_1^2 - X_2^2 + Y_1^2 - Y_2^2 + Z_1^2 - Z_2^2 + \\ &\quad 2X(X_2 - X_1) + 2Y(Y_2 - Y_1) + 2Z(Z_2 - Z_1) \\ S_2^2 - S_3^2 &= X_2^2 - X_3^2 + Y_2^2 - Y_3^2 + Z_2^2 - Z_3^2 + \\ &\quad 2X(X_3 - X_2) + 2Y(Y_3 - Y_2) + 2Z(Z_3 - Z_2). \end{aligned} \qquad (4\text{-}6)$$

Collecting all the known terms of equation (4-6) to the right hand side and those relating to the unknowns on the left hand side leads to

$$\begin{aligned} 2X(X_2 - X_1) + 2Y(Y_2 - Y_1) + 2Z(Z_2 - Z_1) &= a \\ 2X(X_3 - X_2) + 2Y(Y_3 - Y_2) + 2Z(Z_3 - Z_2) &= b \end{aligned} \qquad (4\text{-}7)$$

with the terms $\{a, b\}$ given by

$$\begin{aligned} a &= S_1^2 - S_2^2 - X_1^2 + X_2^2 - Y_1^2 + Y_2^2 - Z_1^2 + Z_2^2 \\ b &= S_2^2 - S_3^2 - X_2^2 + X_3^2 - Y_2^2 + Y_3^2 - Z_2^2 + Z_3^2. \end{aligned} \qquad (4\text{-}8)$$

The solution of the unknown terms $\{X, Y, Z\}$ now involves solving equation (4-7) which has two equations with three unknowns. To circumvent the problem of having more unknowns than the equations, two of the unknowns are sought in terms of the third unknown (e. g. $X = g(Z), Y = g(Z)$).

We express equation (4-7) in the algebraic form

$$\begin{aligned} a_{02}X + b_{02}Y + c_{02}Z + f_{02} &= 0 \\ a_{12}X + b_{12}Y + c_{12}Z + f_{12} &= 0 \end{aligned} \qquad (4\text{-}9)$$

with the coefficients given as

$$\begin{aligned} &a_{02} = 2(X_1 - X_2), b_{02} = 2(Y_1 - Y_2), c_{02} = 2(Z_1 - Z_2) \\ &a_{12} = 2(X_2 - X_3), b_{12} = 2(Y_2 - Y_3), c_{12} = 2(Z_2 - Z_3) \\ &f_{02} = (S_1^2 - X_1^2 - Y_1^2 - Z_1^2) - (S_2^2 - X_2^2 - Y_2^2 - Z_2^2) \\ &f_{12} = (S_2^2 - X_2^2 - Y_2^2 - Z_2^2) - (S_3^2 - X_3^2 - Y_3^2 - Z_3^2). \end{aligned} \qquad (4\text{-}10)$$

The *Groebner basis* is then obtained using the *GroebnerBasis* command in Mathematica 3.0 as

$$\begin{aligned} &GroebnerBasis[\{a_{02}X + b_{02}Y + c_{02}Z + f_{02}, \\ &a_{12}X + b_{12}Y + c_{12}Z + f_{12}\}, \{X, Y\}] \end{aligned} \qquad (4\text{-}11)$$

giving the computed *Groebner basis*

$$\begin{aligned} g_1 &= a_{02}b_{12}Y - a_{12}b_{02}Y - a_{12}c_{02}Z + a_{02}c_{12}Z \\ &\quad + a_{02}f_{12} - a_{12}f_{02} \\ g_2 &= a_{12}X + b_{12}Y + c_{12}Z + f_{12} \\ g_3 &= a_{02}X + b_{02}Y + c_{02}Z + f_{02}. \end{aligned} \qquad (4\text{-}12)$$

The first equation of (4-12) is solved for $Y = g(Z)$ giving

$$Y = \frac{\{(a_{12}c_{02} - a_{02}c_{12})Z + a_{12}f_{02} - a_{02}f_{12}\}}{(a_{02}b_{12} - a_{12}b_{02})} \tag{4-13}$$

which is substituted in the second equation of (4-12) to give $X = g(Y, Z)$ as

$$X = \frac{-(b_{12}Y + c_{12}Z + f_{12})}{a_{12}}. \tag{4-14}$$

The obtained values of Y and X are substituted in the first equation of (4-4) to give a quadratic equation in Z. Once this quadratic has been solved for Z, the values of Y and X can be obtained from (4-13) and (4-14) respectively. We mention here that the direct solution of $X = g(Z)$ could be obtained by computing the *reduced Groebner basis* to give

$$X = \frac{\{(b_{02}c_{12} - b_{12}c_{02})Z + b_{02}f_{12} - b_{12}f_{02}\}}{(a_{02}b_{12} - a_{12}b_{02})} \tag{4-15}$$

rather than solving for $Y = g(Z)$ first and then substituting in the second equation of (4-12) to give $X = g(Z)$ presented in (4-14). Similarly, we could obtain $Y = g(Z)$ alone by replacing Y with X in the option section of the *reduced Groebner basis* discussed in Awange (2002).

Tab. 2: Observations of type space angles

Observation from	Space angle (gon)
K1-Schlossplatz-Liederhalle ψ_{12}	35.84592
K1-Liederhalle-Eduardpfeiffer ψ_{23}	49.66335
K1-Eduardpfeiffer-Schlossplatz ψ_{31}	14.19472

5 Example

Using the computed *univariate polynomials (element of Groebner basis* of the Ideal $I \subset \mathbb{R}[x_1, x_2, x_3]$) in (4-3) in Section 4, we determine the distances $S_i = x_i \in \mathbb{R}^+$, $i = \{1, 2, 3\} \in \mathbb{Z}^3_+$ between the unknown station $P \in \mathbb{E}^3$ and the known stations $P_i \in \mathbb{E}^3$ for the test network *»Stuttgart Central«* in Awange (2002a). The unknown point P_0 in this case is the pillar K1 on top of the University building at Kepler Strasse 11. Points P_1, P_2, P_3 of the tetrahedron $\{P_0P_1P_2P_3\}$ in Fig. 1 correspond to the chosen known GPS stations *Schlossplatz*, *Liederhalle*, and *Eduardpfeiffer*. The distance from *K1 to Schlossplatz* is designated $S_1 = x_1 \in \mathbb{R}^+$, *K1 to Liederhalle* $S_2 = x_2 \in \mathbb{R}^+$, while that of *K1 to Eduardpfeiffer* is designated $S_3 = x_3 \in \mathbb{R}^+$. The distances between the known stations $\{S_{12}, S_{23}, S_{31}\} \in \mathbb{R}^+$ are computed from their respective GPS coordinates in Tab. 1. Their corresponding space angles $\psi_{12}, \psi_{23}, \psi_{31}$ are as given in Tab. 2. From the computed *reduced Groebner basis univariate polynomials* in (4-3), x_1, x_2 and x_3 each have four real roots as indicated in Fig. 2, Fig. 3, and Fig. 4 respectively. The desired distances selected with the help of a priori information are $S_1 = 566.8635$ m, $S_2 = 430.5286$ m, and $S_3 = 542.2609$ m. These values compare well with their real values (e. g. Awange 2002a).

Once the distances have been established, the position is then determined in step 2 via ranging (Awange et al. (2003)) as $X(m) = 4157066.1116$, $Y(m) = 671429.6655$ and $Z(m) = 4774879.3704$ in the Global Reference Frame. The critical configuration of the three-dimensional ranging problem is presented in Awange et al. (2003).

6 Conclusion

We have demonstrated the power of the algebraic computational tool (*reduced Groebner basis*) in solving the problem of *three-dimensional intersection*. We have succeeded in demonstrating that by converting the *nonlinear observation equations* of the *three-dimensional intersection* into algebraic (polynomials), the *multivariate* system of polynomial equations relating the unknown variables (indeterminate) to the known variables can be reduced into a system of polynomial equations consisting of a *univariate polynomial*. We have therefore managed to provide *symbolic solutions* to the *triple three-dimensional intersection* problem by obtaining the *univariate polynomials* that can readily be solved numerically once the observations are available.

Tab. 1: GPS Coordinates in the *Global Reference Frame* $\mathbb{F}^\bullet$ $(X, Y, Z), (X_i, Y_i, Z_i), i = 1, 2, 3$

Station Name	$X(m)$	$Y(m)$	$Z(m)$	$\sigma_X(m)$	$\sigma_Y(m)$	$\sigma_Z(m)$
Schlossplatz	4157246.5346	671877.0281	4774581.6314	0.0008	0.0008	0.0008
Liederhalle	4157266.6181	671099.1577	4774689.8536	0.00129	0.00128	0.00134
Eduardpfeiffer	4156748.6829	671171.9385	4775235.5483	0.00193	0.00184	0.00187

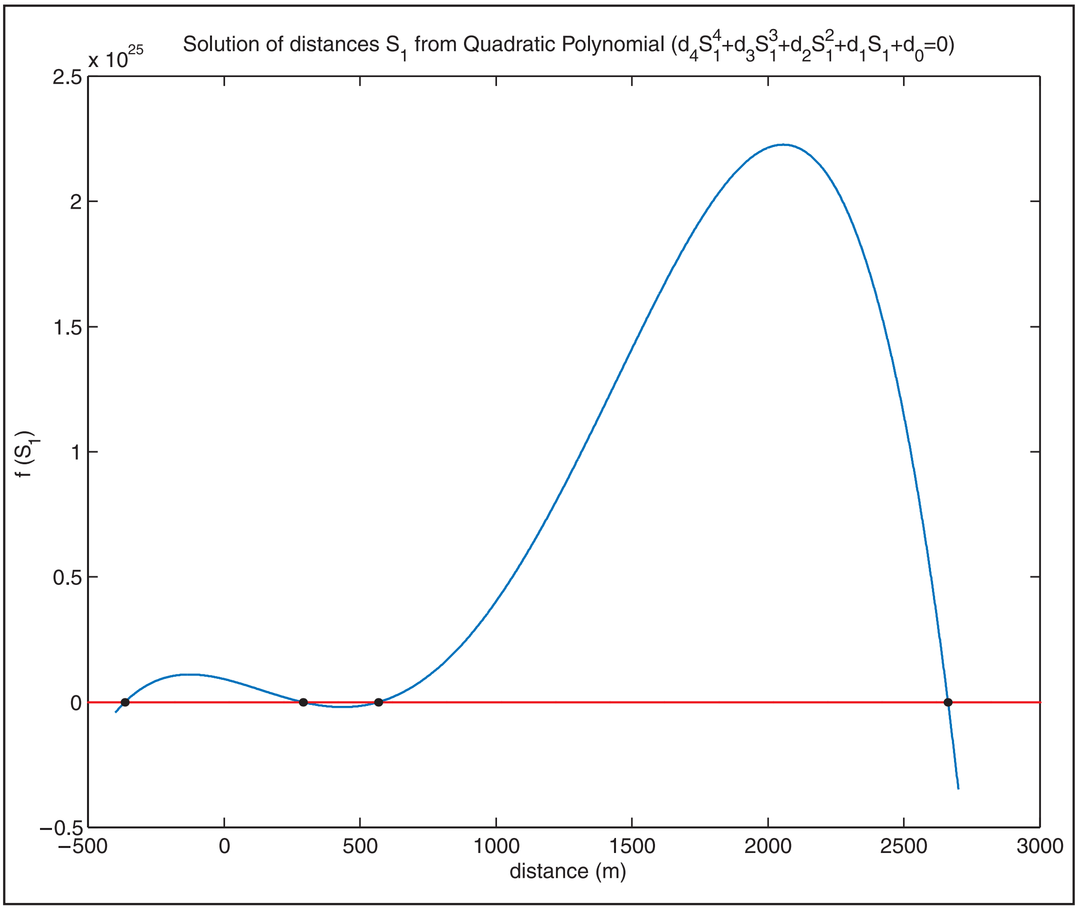

Fig. 2: Solution for distance S_1

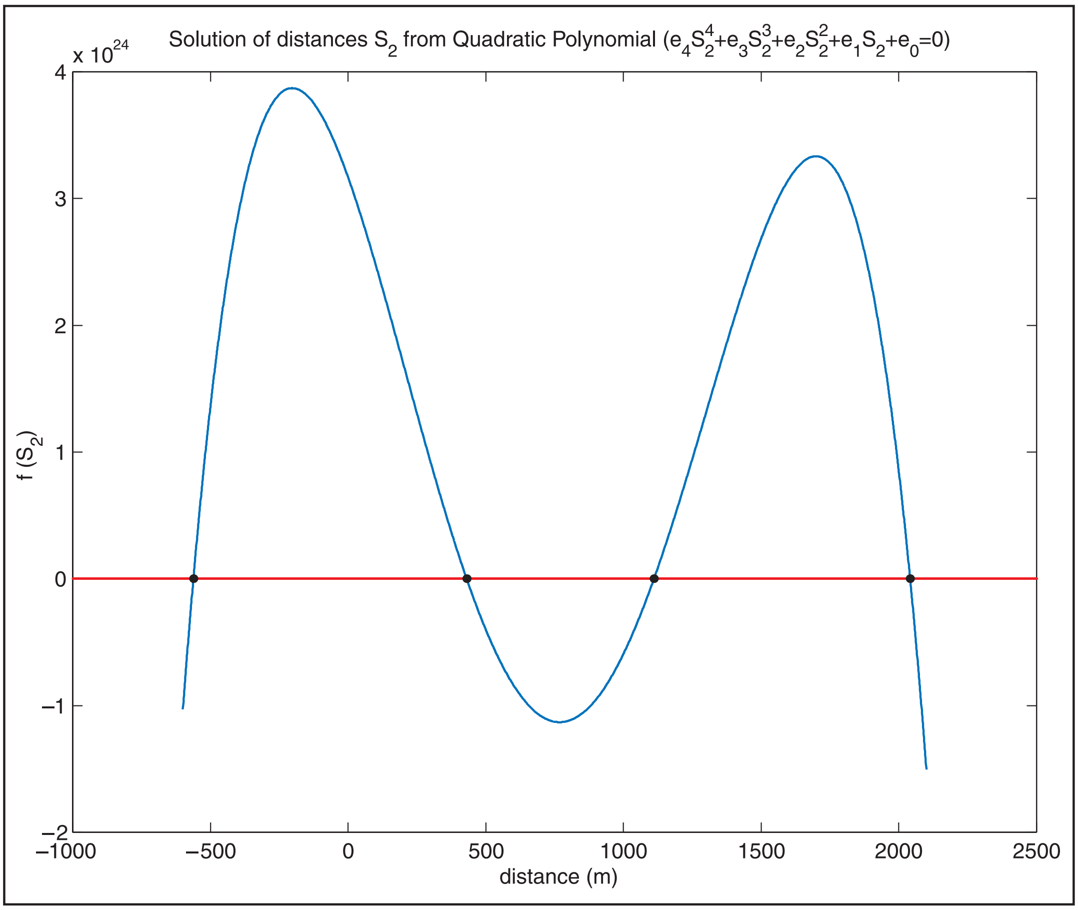

Fig. 3: Solution for distance S_2

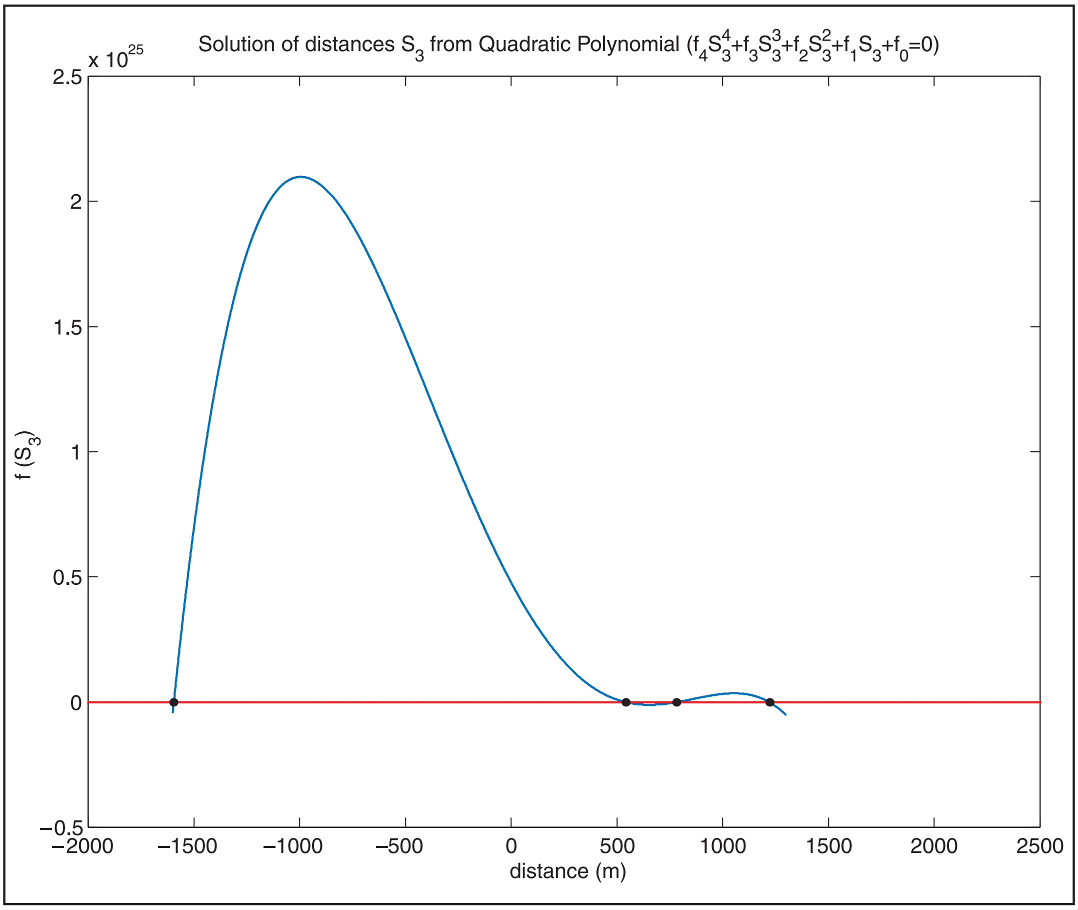

Fig. 4: Solution for distance S_3

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Authors' addresses

Dr.-Ing. Joseph L. Awange
Department of Geophysics, Kyoto University
Kitashirakawa Oiwake-cho, Sakyo-ku
Kyoto City, Kyoto 606-8502, Japan
Tel: +81-75-753-3319, awange@kugi.kyoto-u.ac.jp

Prof. Dr.-Ing. habil. Dr. h. c. mult. Erik W. Grafarend
Department of Geodesy and GeoInformatics, University of Stuttgart
Geschwister-Scholl-Str. 24 D, 70174 Stuttgart, Germany
Tel: +49-711-1213389, grafarend@gis.uni-stuttgart.de

Prof. Yoichi Fukuda
Department of Geophysics, Kyoto University
Kitashirakawa Oiwake-cho, Sakyo-ku
Kyoto City, Kyoto 606-8502, Japan
fukuda@kugi.kyoto-u.ac.jp

Appendix: Coefficients of Quartic polynomials

(Eq. 4-3)

$$d_4 = (-2b_1^2b_2^2 + b_3^4 + b_1^4 - 2b_1^2b_3^2 + b_2^4 - 2b_2^2b_3^2)$$

$$d_3 = (-4a_0b_1b_3^2 - 2b_1^3b_2^2 + 4b_1^3c_0 - 4a_0b_1b_2^2 + 2b_1b_3^4 + 4b_0b_1^3 - 2b_1^3b_3^2 + 2b_1b_2^4 + 4a_0b_1^3 - 4b_0b_1b_3^2 - 4b_1b_2^2c_0 - 4b_1b_3^2c_0 - 4b_0b_1b_2^2)$$

$$d_2 = (-4b_0b_1^2b_2^2 - 4b_0b_2^2c_0 + 2b_1^2b_2^2b_3^2 - 2b_0^2b_2^2 - 2a_0^2b_2^2 - 4b_1^2b_3^2c_0 - 4a_0b_0b_3^2 + 4b_2^2b_3^2c_0 + 12b_0b_1^2c_0 + 12a_0b_1^2c_0 - 6a_0b_1^2b_3^2 + 12a_0b_0b_1^2 - 4a_0b_0b_2^2 - 4b_0b_3^2c_0 + 2b_0b_3^4 - 2b_3^2c_0^2 + 6b_1^2c_0^2 + 2a_0b_3^4 - 2a_0^2b_3^2 - 4a_0b_2^2c_0 - 2b_0^2b_3^2 - b_2^2b_3^4 + 2a_0b_2^4 - 4b_1^2b_2^2c_0 - 6b_0b_1^2b_3^2 + 6a_0^2b_1^2 + 2b_0b_2^2b_3^2 - 4a_0b_3^2c_0 - 6a_0b_1^2b_2^2 + b_1^2b_2^4 + 6b_0^2b_1^2 + b_1^2b_3^4 - 2b_2^2c_0^2)$$

$$d_1 = (2a_0b_1b_3^4 - 6b_0^2b_1b_3^2 - 2b_1b_2^2c_0^2 - b_1b_2^2b_3^4 + 4b_1c_0^3 + 12b_0^2b_1c_0 + 4b_0^3b_1 - 2b_0^2b_1b_2^2 - 6a_0^2b_1b_3^2 + 2a_0b_1b_2^4 + 2b_0b_1b_3^4 + 12a_0b_1c_0^2 - 6a_0^2b_1b_2^2 + 12a_0^2b_1c_0 + 4a_0^3b_1 + 12b_0b_1c_0^2 + 12a_0^2b_0b_1 - 2b_1b_3^2c_0^2 + 12a_0b_0^2b_1 - 8a_0b_1b_3^2c_0 - 8a_0b_1b_2^2c_0 + 24a_0b_0b_1c_0 - 4b_0b_1b_2^2c_0 - 8b_0b_1b_3^2c_0 - 12a_0b_0b_1b_3^2 - 8a_0b_0b_1b_2^2 + 2b_0b_1b_2^2b_3^2 + 4a_0b_1b_2^2b_3^2 + 4b_1b_2^2b_3^2c_0)$$

$$d_0 = -a_0b_2^2b_3^4 + a_0^2b_2^4 - 4a_0^2b_0b_2^2 + 12a_0^2b_0c_0 + 2a_0^2b_2^2b_3^2 + 12a_0b_0^2c_0 + c_0^4 + b_0^2b_3^4 + a_0^4 + a_0^2b_3^4 - 2a_0^3b_2^2 - 2a_0b_3^2c_0^2 - 4a_0^2b_3^2c_0 + 4a_0c_0^3 - 6a_0b_0^2b_3^2 + 6a_0^2b_0^2 - 2b_0^3b_3^2 - 4a_0^2b_2^2c_0 + 4b_0^3c_0 + 4a_0b_0^3 + b_0^4 + 4b_0c_0^3 - 2a_0b_2^2c_0^2 - 6a_0^2b_0b_3^2 + 4a_0^3b_0 + 12a_0b_0c_0^2 - 2b_0b_3^2c_0^2 + 6b_0^2c_0^2 + 4a_0^3c_0 + 6a_0^2c_0^2 - 4b_0^2b_3^2c_0 + 2a_0b_0b_3^4 - 2a_0b_0^2b_2^2 - 4a_0b_0b_2^2c_0 - 8a_0b_0b_3^2c_0 + 2a_0b_0b_2^2b_3^2 + 4a_0b_2^2b_3^2c_0 - 2a_0^3b_3^2$$

$$e_4 = (-2b_1^2b_3^2 + b_1^4 + b_3^4 + b_2^4 - 2b_2^2b_3^2 - 2b_1^2b_2^2)$$

$$e_3 = (-2b_1^2b_2^3 - 4b_0b_2b_3^2 + 4b_2^3c_0 - 4a_0b_2b_3^2 + 2b_1^4b_2 - 2b_2^3b_3^2 - 4a_0b_1^2b_2 - 4b_1^2b_2c_0 - 4b_0b_1^2b_2 + 2b_2b_3^4 + 4b_0b_2^3 + 4a_0b_2^3 - 4b_2b_3^2c_0)$$

$$e_2 = (-4b_0b_1^2c_0 - 4a_0b_2^2b_3^2 + 12a_0b_2^2c_0 - 6b_0b_2^2b_3^2 - 2b_3^2c_0^2 + 2b_0b_1^4 + 6a_0^2b_2^2 - b_1^4b_3^2 + 6b_0^2b_2^2 - 2a_0^2b_3^2 + 12b_0b_2^2c_0 - 4b_2^2b_3^2c_0 - 4b_0b_3^2c_0 + 4a_0b_1^2b_3^2 + 2b_0b_3^4 - 4a_0b_1^2b_2^2 - 6b_0b_1^2b_2^2 + b_1^4b_2^2 - 4a_0b_3^2c_0 + 12a_0b_0b_2^2 - 4a_0b_0b_3^2 + 2b_1^4c_0 - 2a_0^2b_1^2 - 2b_1^2c_0^2 + b_2^2b_3^4 - 2b_0^2b_3^2 - 2b_0^2b_1^2 + 6b_2^2c_0^2 - 6b_1^2b_2^2c_0 + 2b_1^2b_2^2b_3^2 + 2b_1^2b_3^2c_0 - 4a_0b_1^2c_0 - 4a_0b_0b_1^2)$$

$$e_1 = (12b_0^2b_2c_0 - 6b_0^2b_2b_3^2 - b_1^4b_2b_3^2 + 4b_0b_1^2b_2b_3^2 + 24a_0b_0b_2c_0 - 4a_0b_2b_3^2c_0 - 8b_0b_2b_3^2c_0 + 2b_1^2b_2b_3^2c_0 + 4a_0b_1^2b_2b_3^2 - 8a_0b_1^2b_2c_0 - 8a_0b_0b_2b_3^2 - 8a_0b_0b_1^2b_2 - 12b_0b_1^2b_2c_0 + 4a_0^3b_2 + 4b_0^3b_2 + 12a_0^2b_2c_0 - 6b_1^2b_2c_0^2 + 12a_0^2b_0b_2 - 6b_0^2b_1^2b_2 + 2b_0b_2b_3^4 - 2a_0^2b_1^2b_2 - 2b_2b_3^2c_0^2 + 2b_1^4b_2c_0 + 12a_0b_2c_0^2 + 2b_0b_1^4b_2 + 12a_0b_0^2b_2 - 2a_0^2b_2b_3^2 + 12b_0b_2c_0^2 + 4b_2c_0^3)$$

$$e_0 = -b_0b_1^4b_3^2 + c_0^4 - 2a_0^2b_1^2c_0 + 12a_0b_0c_0^2 - 2a_0^2b_0b_1^2 + 4b_0^3c_0 + b_0^4 + b_1^4c_0^2 - 2b_0^3b_3^2 + a_0^4 - 2b_0^3b_1^2 + b_0^2b_1^4 + b_0^2b_3^4 + 2b_0^2b_1^2b_3^2 + 6a_0^2b_0^2 + 4a_0c_0^3 - 2a_0^2b_0b_3^2 - 4a_0b_0^2b_1^2 + 4a_0b_0^3 + 6b_0^2c_0^2 - 4a_0b_0^2b_3^2 - 2b_0b_3^2c_0^2 + 4b_0c_0^3 + 2b_0b_1^4c_0 + 12a_0^2b_0c_0 - 4a_0b_1^2c_0^2 - 6b_0^2b_1^2c_0 - 4b_0^2b_3^2c_0 + 12a_0b_0^2c_0 + 6a_0^2c_0^2 - 6b_0b_1^2c_0^2 + 4a_0^3c_0 - 2b_1^2c_0^3 + 4a_0^3b_0 - 8a_0b_0b_1^2c_0 + 4a_0b_0b_1^2b_3^2 - 4a_0b_0b_3^2c_0 + 2b_0b_1^2b_3^2c_0$$

$$f_4 = (-2b_1^2b_3^2 + b_1^4 + b_3^4 + b_2^4 - 2b_2^2b_3^2 - 2b_1^2b_2^2)$$

$$f_3 = (-4a_0b_2^2b_3 - 4a_0b_1^2b_3 + 4b_3^3c_0 - 2b_1^2b_3^3 + 2b_1^4b_3 + 2b_2^4b_3 + 4b_0b_3^3 - 4b_0b_1^2b_3 - 4b_1^2b_3c_0 + 4a_0b_3^3 - 2b_2^2b_3^3 - 4b_2^2b_3c_0 - 4b_0b_2^2b_3)$$

$$f_2 = (-4b_0b_1^2c_0 - 6a_0b_2^2b_3^2 - 4a_0b_2^2c_0 - 4b_0b_2^2b_3^2 + 6b_3^2c_0^2 - 2a_0^2b_2^2 + b_1^4b_3^2 - 2b_0^2b_2^2 + 6a_0^2b_3^2 - 4b_0b_2^2c_0 - 4b_0b_1^2b_3^2 - 6b_2^2b_3^2c_0 + 12b_0b_3^2c_0 - 4a_0b_1^2b_3^2 + 2a_0b_1^2b_2^2 + 4b_0b_1^2b_2^2 + 12a_0b_3^2c_0 - 4a_0b_0b_2^2 + 12a_0b_0b_3^2 + 2b_1^4c_0 - 2a_0^2b_1^2 - 2b_1^2c_0^2 + 6b_0^2b_3^2 - 2b_0^2b_1^2 - 2b_2^2c_0^2 + 2b_1^2b_2^2b_3^2 - 6b_1^2b_3^2c_0 - 4a_0b_1^2c_0 - 4a_0b_0b_1^2 + b_2^4b_3^2 - b_1^2b_2^4 + 2a_0b_2^4 + 2b_2^4c_0)$$

$$f_1 = (-6a_0^2b_2^2b_3 + 2a_0b_2^4b_3 + 2b_2^4b_3c_0 + 12b_0^2b_3c_0 + 2b_1^4b_3c_0 - b_1^2b_2^4b_3 + 12b_0b_3c_0^2 - 2b_0^2b_1^2b_3 + 12a_0^2b_3c_0 + 12a_0b_0^2b_3 - 6b_2^2b_3c_0^2 - 2b_0^2b_2^2b_3 - 6b_1^2b_3c_0^2 + 4b_0^3b_3 + 4a_0^3b_3 + 4b_3c_0^3 + 12a_0^2b_0b_3 + 12a_0b_3c_0^2 - 2a_0^2b_1^2b_3 + 4b_0b_1^2b_2^2b_3 - 8a_0b_0b_2^2b_3 - 8a_0b_1^2b_3c_0 - 8b_0b_2^2b_3c_0 - 4a_0b_0b_1^2b_3 - 12a_0b_2^2b_3c_0 - 8b_0b_1^2b_3c_0 + 2a_0b_1^2b_2^2b_3 + 24a_0b_0b_3c_0 + 4b_1^2b_2^2b_3c_0)$$

$$f_0 = -b_1^2b_2^4c_0 + 2a_0b_2^4c_0 + c_0^4 - 2a_0^2b_1^2c_0 + 12a_0b_0c_0^2 + 4b_0^3c_0 + b_0^4 + b_1^4c_0^2 + b_2^4c_0^2 + a_0^2b_2^4 + a_0^4 - 6a_0^2b_2^2c_0 + 6a_0^2b_0^2 - 2b_2^2c_0^3 - 4b_0b_2^2c_0^2 + 4a_0c_0^3 + 4a_0b_0^3 + 6b_0^2c_0^2 + 4b_0c_0^3 - 4a_0^2b_0b_2^2 + 12a_0^2b_0c_0 - 4a_0b_1^2c_0^2 - 2a_0b_0^2b_2^2 + 2b_1^2b_2^2c_0^2 - 2b_0^2b_1^2c_0 - 6a_0b_2^2c_0^2 + 12a_0b_0^2c_0 + 6a_0^2c_0^2 - 4b_0b_1^2c_0^2 + 4a_0^3c_0 - 2a_0^3b_2^2 - 2b_1^2c_0^3 + 4a_0^3b_0 - 8a_0b_0b_2^2c_0 - 4a_0b_0b_1^2c_0 + 2a_0b_1^2b_2^2c_0 - 2b_0^2b_2^2c_0 + 4b_0b_1^2b_2^2c_0$$