Polynomial optimization of the 7-parameter datum transformation problem when only three stations in both systems are given

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Summary

We present here the *Gauss-Jacobi combinatorial* algorithm to solve in a closed form the overdetermined problem of *7-parameter datum transformation* with only bare minimum number of points (i.e. three points in both coordinate systems). From the **nine** *7-parameter datum transformation* equations, 36 minimum combinatorial subsets, each comprising seven equations are formed and solved using the *Groebner basis* algorithm in the *first* step. With each minimal combinatorial subset yielding seven elements of the solution set, a total of $\{7 \times 36 = 252\}$ solutions are formed. The 252 *minimum combinatorial solutions* are reduced to their final adjusted values in step *two* by means of their weighted mean via the *nonlinear error/variance-covariance propagation*. The advantage is that the *Groebner basis algorithm* (the computing engine of the *Gauss-Jacobi combinatorial algorithm*), which is already implemented in algebraic software such as *Mathematica* and *Maple*, does not require approximate starting values, as is always the case with traditional procedures (iterative/linearization). The procedure makes it possible for the stochasticity of both coordinate systems involved to be taken into account and becomes handy in a situation where only minimum points are given with no knowledge of the initial approximate values.

Zusammenfassung

Der Gauß-Jacobi Kombinatorik-Algorithmus *ist in der Lage, eine Datumtransformation [7-Parameter Ähnlichkeitstransformation, konforme Gruppe $C_7(3)$] auch im überbestimmten Fall in geschlossener Form – ohne Linearisierung – als Lösung darzustellen. Hier widmen wir uns dem* klassischen Fall, *aus den gegebenen Koordinaten* dreier Punkte *im alten und neuen Koordinatensystem die sieben Parameter einer Ähnlichkeitstransformation (drei Parameter der Translation, drei Parameter der Rotation, ein Parameter des Maßstabes, auch Dilatation genannt)* ohne Linearisierung *des Gleichungssystems zu bestimmen:* Neun *nichtlineare Gleichungen stehen* sieben *unbekannten Datumparametern gegenüber. 36* Gauß-Jacobi Kombinationslösungen *werden präsentiert, die mittels des* Gröbner Basis Algorithmus im ersten Schritt *bestimmt werden. Aus jeder Untermenge der Lösungen werden* $\{7 \times 36 = 252\}$ *Lösungen konstruiert, die im* zweiten Schritt *auf eine Lösung als gewogenes Mittel der Einzellösungen, (Ausgleichung direkter Pseudo-Beobachtungen, L_2-Norm Approximation) reduziert werden. Das Fehlerfortpflanzungsgesetz liefert die Varianz-Kovarianz Matrix der Einzellösungen, ihre Inverse die Gewichtsmatrix für das gewogene Mittel (»Baryzentrum«). Der methodische Vorteil der vorgelegten Lösung einer* Helmert-Transformation *besteht im Folgenden: Die algebraische »Software« vom Typ* Mathematica *und* Maple *zur Lösung algebraischer Gleichungen (polynomische Gleichungssysteme) benötigt* keine Näherungswerte *und* keine Iterationen*. Die Varianz-Kovarianz-Matrizen alter und neuer Koordinaten finden vollständige Berücksichtigung.*

1 Introduction

In our extended coverage of the 7-parameter datum transformation problem, we started in Awange and Grafarend (2003) by presenting the *Groebner basis* algebraic procedure that could be used to solve explicitly the 7-parameter datum transformation problem for the 3-rotation, 3-translation and 1-scale elements. It was demonstrated that given three points in two systems, one could obtain with *Groebner basis* the $C_7(3)$ 7-transformation parameters without the need for either **linearization** or **iteration**. The approach required that the *7-parameter datum transformation problem* equations first be converted into its algebraic (polynomial) form. In particular, the algebraic tool of *Groebner basis* provided a symbolic solution to the *7-parameter datum transformation problem* and showed that the scale parameter fulfils a *quartic polynomial* (*univariate* polynomial of degree four) while the rotation parameters were given by the linear functions once the scale parameter had been solved (see also Awange 2002a, Boxes 5-3, 5-4 p. 89). The translation parameters were eliminated at the differencing stage (equations 5-18, 5-19 in Awange 2002a, p. 88). They are only solved once the scale and rotation elements have been obtained.

In the approach above, the *Groebner basis* operated in the manner similar to the *Gauss elimination technique* used for solving linear system of equations. The only difference is that Groebner *basis* was subjected to *nonlinear system* of equations where it was used to solve several variables in a multivariate system as opposed to the *Gauss-elimination* technique. The end product of the *Groebner basis approach* (with the *lexicographic ordering* of the monomials chosen) consisted of a **univariate polynomial** equation whose roots were obtained using the roots command of *MATLAB* software.

In practice however, one is usually compounded by an overdetermined case where more observations are measured than the bare minimum required to solve for the unknowns. In the *Groebner basis* algorithm in Awange and Grafarend (2002a) discussed above, only minimum information necessary to solve the 7-parameter datum transformation problem $C_7(3)$ was used. This involved choosing **seven** from the usual **nine** equations involved in the *7-parameter transformation problem*. In the present contribution, we extend on the contribution of Awange and Grafarend (2002a) by considering all nine equations but with only the bare minimum number of points required, i. e. three stations in both systems. In essence, we solve here the overdetermined problem in a closed form. Forming combinations here referred to as combinatorials from the nine equations of the 7-parameter datum transformation problem and solving each combinatorial solution in a closed form via the *Groebner basis algorithm* achieve this. This particular approach becomes vital when one has only the bare minimum information (three points in both coordinate systems) and all the information have to be used to determine the *7-parameter datum transformation problem* $C_7(3)$. Given the nine nonlinear equations formed using the three stations in two systems, combinatorials are formed each containing 7-nonlinear equations, which are solved using the *Groebner basis algorithm*. The obtained combinatorial solutions form the pseudo-observations whose variance-covariance matrix has to be obtained via the *nonlinear variance-covariance/error propagation* in the second step. The final step involves the adjustment by using the *special linear Gauss-Markov model*, as the pseudo-observations are linearly independent entities.

Once the 7-parameter datum transformations problem has been solved and the parameters obtained, they are used to transform coordinates from a local system to the WGS 84 system. The *Groebner basis algorithm* employed to solve the minimal combinatorial subsets makes use of the *skew-symmetric* matrix to construct the orthogonal matrix X_3. Other approaches for parametrizing the rotation matrix have been presented e. g. in Shut (1958/59) and Thompson (1959a, b).

Whereas here we look at the case where only the bare minimum number of stations are involved, we refer the curious reader to our sister paper Awange and Grafarend (2002b) for the case where more than three points in both systems has been solved. In that contribution, Awange and Grafarend (2002b) compared the overdetermined solution of the nonlinear 7-parameter datum transformation problem using the *Gauss-Jacobi combinatorial algorithm* and *linearized least squares* procedures. The *Gauss-Jacobi combinatorial algorithm* was found to have the following advantages:

1. From the start, the objective was known.
2. The approach did not require linearization.
3. The need for iteration did not exist.
4. The variance-covariance matrix of both coordinate systems were considered automatically and
5. Outliers became visible.

The difference between the approach being considered in the present contribution and that of Awange and Grafarend (2002b) is that in the previous paper, combinatorials were formed from the **number of common points** in two stations (seven points) to derive the expressions in Boxes (2-2) and (2-3). In the present contribution however, the number of common points in the two systems (three points) are held constant while the **nine equations** formed using the three points are used to form the combinatorial sets. In Grafarend and Awange (2003), a different approach of the *weighted Procrustes algorithm* is applied to solve the overdetermined *7-parameter transformation problem* in a closed form.

We organize the present contribution as follows; in Section 2, we present an outline of the *Gauss-Jacobi combinatorial algorithm*. The mathematical concepts underlying the procedure have been omitted and instead we refer to Awange (2002a, b). The section only presents the operational mode of the algorithm. Section 3 considers the use of the algorithm to solve the 7-parameter datum transformation problem while Section 4 considers a case study before concluding in Section 5.

2 The Gauss-Jacobi combinatorial algorithm

The *Gauss-Jacobi combinatorial algorithm* named after *C. F. Gauss* (see *Appendix* A-4, pp. 110–112 in Awange 2002a) and *C. G. I. Jacobi* operate in *three phases*. In the *first phase*, one forms minimal combinations from the observation sample that can be solved in a closed form to obtain the desired solutions. The net result is that one ends up with pseudo-observations, which are within the solution space of the desired values. This *first phase* in essence projects a **nonlinear** case into **linear** case. The essential point to remember is that one is starting with a **nonlinear system of equations**. The process of solving the minimal combinatorial subsets is a kin to the *Gauss-elimination* technique used for solving linear system of equations.

Once the minimal combinatorial subsets have been formed, one employs the *Groebner basis* algebraic technique to solve the nonlinear system of equations. This algebraic technique exists already in algebraic software such as *Matlab* and *Maple*. The theory of *Groebner basis* has been outlined in Awange (2002) and has been elaborately presented in textbooks such as Cox et al. (1997, 1998).

Once the *first phase* is successfully carried out with the solutions of the various subsets acting as pseudo-observations, the *nonlinear variance-covariance/error propagation* has to be carried out in the *second phase* to obtain the weight matrix of the pseudo-observations. This then requires that the stochasticity of the initial observational sample be known in order to propagate these to the pseudo-observations.

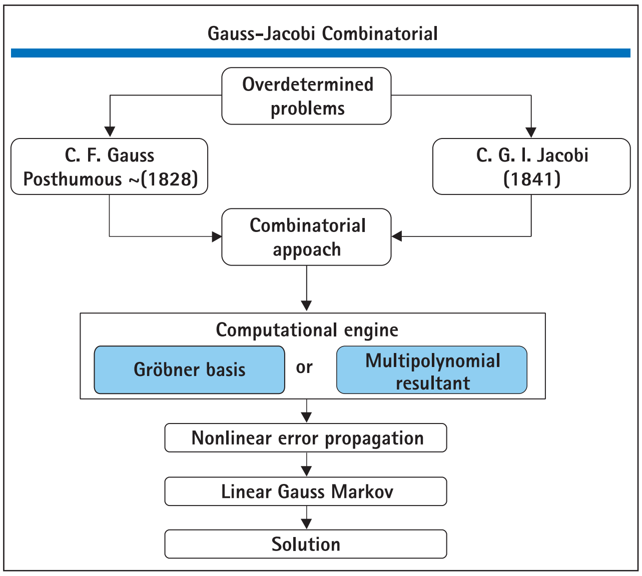

Fig. 1: The Gauss-Jacobi combinatorial algorithm

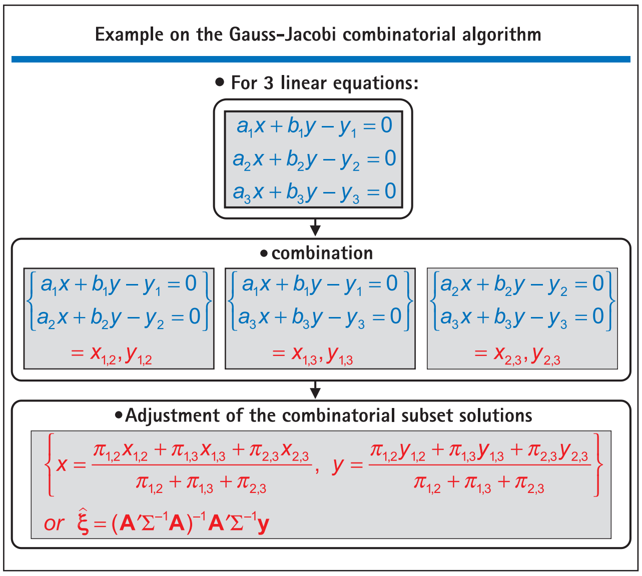

Fig. 2: Example of application of Gauss-Jacobi algorithm

The *final phase* entails the adjustment step, which is performed to obtain the adjusted value. Since the pseudo-observations are linearly independent, the *special linear Gauss-Markov model* (Awange 2002a, Definition 2-0a, p. 7) is employed. In order to understand how the program works, we present the schematic diagram of the program in *Fig. 1* and that of an example in *Fig.* 2.

From Fig. 1, the program operates in the following steps.

- **Step 1:** Given an overdetermined system with n observations in m unknowns, form from the n observations the

$$k(no\ of\ combinations) = \frac{n!}{m!(n-m)!} \tag{2-1}$$

  minimal combination that comprise m equations that are to be solved in closed form using the *Groebner basis* algebraic technique.
- **Step 2:** Solve each set of m equations from *Step 1* above using either *Groebner basis* algebraic technique.
- **Step 3:** Perform the *nonlinear error/variance-covariance* propagation to obtain the variance-covariance matrix of the pseudo-observations obtained in *Step* 2.
- **Step 4:** Using the pseudo-observations of *Step* 2 and the variance-covariance matrix from *Step 3*, adjust the pseudo-observations via the *special linear Gauss-Markov model.*

The following example based on a linear case illustrates the principles behind the algorithm.

Example: From *Fig.* 2, consider a case where three linear equations have been given for the purpose of solving the two unknowns (x, y). Three possible combinations each containing two equations necessary for solving the two unknowns can be formed as shown in the box labelled **combination**. Each of the system of two linear equations is either solved by substitution, graphically or matrix form to give **three pairs** of solutions $\{x_{1,2}, y_{1,2}\}$, $\{x_{2,3}, y_{2,3}\}$, $\{x_{1,3}, y_{1,3}\}$. The final step now involves the adjustment of these pseudo-observations $\{x_{1,2}, y_{1,2}, x_{2,3}, y_{2,3}, x_{1,3}, y_{1,3}\}$ as indicated in the box **adjustment** of the **combinatorial subsets solutions** with the weight matrix $\mathbf{\Sigma}^{-1}$ respectively weight elements $\{\pi_{1,2}, \pi_{2,3}, \pi_{1,3}\}$ obtained via nonlinear *error/variance-covariance* propagation.

3 Algebraic solution of the overdetermined 7-parameter datum transformation problem

In Awange (2002a, *Boxes* 5-3, 5-4, p. 89) and Awange and Grafarend (2002, *Boxes* 2-2, 2-3), the *univariate polynomial* for solving the *scale parameters* and the linear functions (in terms of scale and coordinate differences) for solving the *rotation parameters* were derived using *Groebner basis.* In the current contribution, the *Groebner basis algorithm* is applied to each combinatorial set containing seven equations to solve for the scale and rotation parameters. In total, we have 36 expressions similar to those in Awange (2002, *Boxes* 5-3, 5-4, p. 89) and Awange and Grafarend (2002, *Boxes* 2-2, 2-3), each solving a different set of equations. In Awange and Grafarend (2002) the expressions in the *Boxes* (2-2) and (2-3) were held fixed while the **numbers of points** in both systems were used to form the combinatorials. Here, the numbers of points are held constant while the **equations** are used to form the combinatorials.

Given three points in both systems involved in the closed form transformation process, nine nonlinear equations are formed as in Awange and Grafarend (2002, equations 2-9, p. 4). From the nine nonlinear equations and with (2-1), 36 sets of equations each comprising seven equations are formed and each set solved as discussed in Awange (2002a, pp. 88–89) and Awange and Grafarend (2002). For each set, the expressions for scale and rotation elements are given. Thus we end up with 36 expressions similar to those in Awange (2002a, *Boxes* 5-3, 5-4, p. 89) and Awange and Grafarend (2002, *Boxes* 2-2, 2-3). Expressions from each set are then used together with the three stations in both systems to compute the scale and rotation parameters. Once the scale and the rotation parameters have been computed for each set, they are used to determine the translation elements.

The computed elements are then considered as pseudo-observations and the *special linear Gauss-Markov model* applied to get the barycentric solutions. In order to achieve this, use is made of the *nonlinear error/variance-covariance propagation* (i. e. *equations* 2-10 and 2-11 in Awange and Grafarend 2002) to obtain the variance-covariance matrix of the pseudo-observations.

4 Case Study

We consider Cartesian coordinates of three stations given in the Local and Global Reference Systems (WGS 84) as in *Tables* (1) and (2). Desired are the *7-parameters of datum transformation problem*. Taking the three coordinates in both systems and applying the *Groebner basis algorithm*, the seven transformation parameters are obtained for each minimum combinatorial set. The scatter of the 36 combinatorial solutions for scale, elements of the symmetric matrix X_3 and the translation parameters are plotted around the adjusted values in *Figures 3* and *4*. *Figure 3* indicates the scatter of the computed 36 minimal combinatorial solutions of scale (indicated by dotted points •) around the adjusted values indicated by a line (–). *Figures 4* indicate the scatter of the computed 36 minimal combinatorial solutions of translation and rotation parameters (indicated by dotted points • around the adjusted values indicated by a star (*). *Table 3* presents the final adjusted values of the *7-parameter datum transformation* which are used to transform the Cartesian coordinates from the *Local Reference System (Table 1)* to the *Global Reference System* (WGS 84, *Table* 2) in *Table 4*.

The figures clearly identify the outlying combinations from which the respective (suspected outlying points) points can be deduced. The computed residuals in *Table 4* are in cm range.

5 Conclusion

Given three points in two systems, it has been demonstrated here that the 7-parameter datum transformation can be obtained in an optimal way with the help of *Gauss-Jacobi combinatorial algorithm*. The need for *iteration* and *linearization* is not required except for the *nonlinear error/variance-covariance propagation* when generating the weight matrix needed in the final adjustment step.

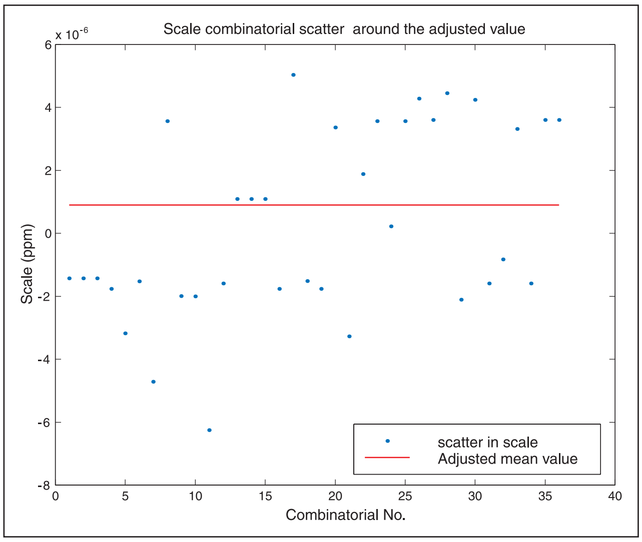

Fig. 3: Scatter of the 36 combinatorial solutions of scale around adjusted value

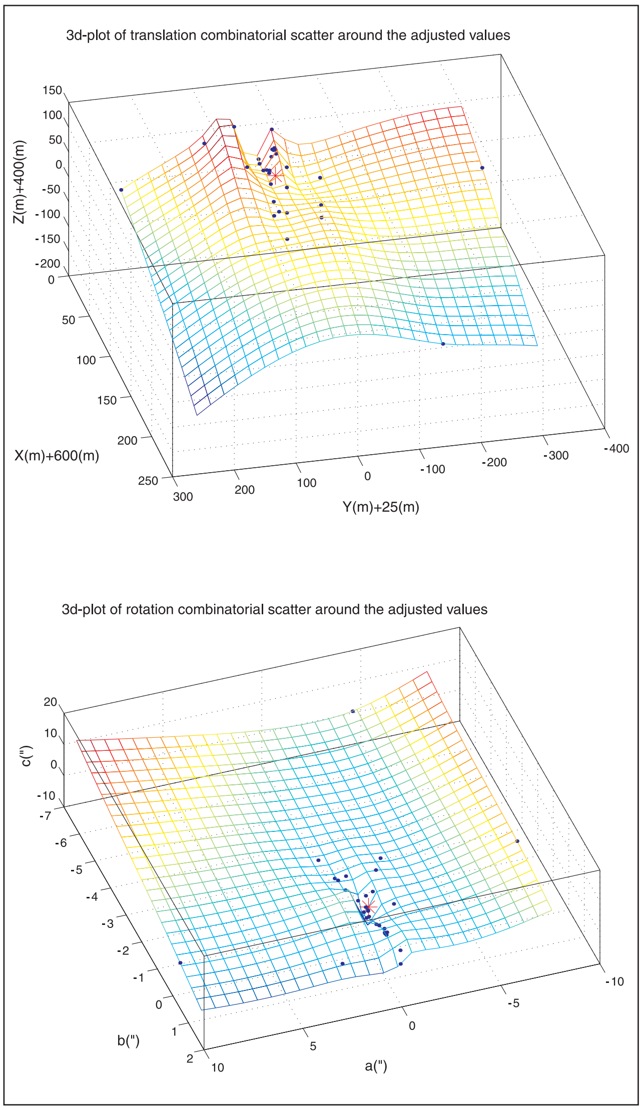

Fig. 4: Scatter of the 36 computed combinatorial solutions of translation and rotation around the adjusted values

Table 1: Coordinates of System A

Station Name	$X(m)$	$Y(m)$	$Z(m)$
Solitude	4157222.543	664789.307	4774952.099
Buoch Zeil	4149043.336	688836.443	4778632.188
Hohenneuffen	4172803.511	690340.078	4758129.701

Table 2: Coordinates of System B

Station Name	$X(m)$	$Y(m)$	$Z(m)$
Solitude	4157870.237	664818.678	4775416.524
Buoch Zeil	4149691.049	688865.785	4779096.588
Hohenneuffen	4173451.354	690369.375	4758594.075

Table 3: Computed 7-parameter using Gauss-Jacobi combinatorial algorithm

Transformation Parameter	Value	Root-mean-square	Unit
Scale $k-1$	0.90492535	0.498669714	$[ppm]$
Rotation X_1 (a)	−0.29496700	0.068980382	["]
Rotation X_2 (b)	−0.52717159	0.072002244	["]
Rotation X_3 (c)	−0.27335444	0.067150234	["]
Translation ΔX	655.2738	4.0518	$[m]$
Translation ΔY	27.4065	4.997	$[m]$
Translation ΔZ	450.4307	3.6986	$[m]$

Table 4: Transformed Cartesian coordinates of System A (Table 1) into System B (Table 2) using the 7-datum transformation parameters of Table 3 computed by Gauss-Jacobi algorithm.

Site	$X(m)$	$Y(m)$	$Z(m)$
System A: Solitude	4157222.5430	664789.3070	4774952.0990
System B	4157870.2370	664818.6780	4775416.5240
Transformed value	4157870.2560	664818.6341	4775416.5250
Residual	**−0.0190**	**0.0439**	**−0.0010**
System A: Buoch Zeil	4149043.3360	688836.4430	4778632.1880
System B	4149691.0490	688865.7850	4779096.5880
Transformed value	4149691.0640	688865.8079	4779096.5621
Residual	**−0.0150**	**−0.0229**	**0.0259**
System A: Hohenneuffen	4172803.5110	690340.0780	4758129.7010
System B	4173451.3540	690369.3750	4758594.0750
Transformed value	4173451.3149	690369.3835	4758594.1151
Residual	**0.0391**	**−0.0085**	**−0.0400**

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