Linearized Least Squares and nonlinear Gauss-Jacobi combinatorial algorithm applied to the 7-parameter datum transformation $\mathbb{C}_7(3)$ problem

Joseph L. Awange and Erik W. Grafarend

Summary

We present here the *linearized Least Squares* and the nonlinear *Gauss-Jacobi combinatorial* solution of the *7-parameter datum transformation* problem. From the coordinates of seven stations in both local and WGS 84 systems, the *7 transformation parameters* are determined and used to transform coordinates from the local system to their corresponding values in the WGS 84 system. The residual are computed by obtaining the difference between the transformed coordinates and their actual values in the WGS 84 system. From these residuals, we compute the norm in order to asses the strength of the procedures. The computed norm from the *Gauss-Jacobi combinatorial* solution is somewhat better than those of the *linearized least squares solution*. The proposed *Gauss-Jacobi combinatorial* solution offers the following advantages: (i) from the start, the objective is known (ii) the approach does not require linearization (iii) the need for iteration does not exist (iv) the variance-covariance matrices of both coordinate systems can be considered automatically (v) outliers become visible.

Zusammenfassung

Die 7-parametrige Datumtransformation untersuchen wir mittels der linearisierten Methode der kleinsten Quadrate *im Vergleich zur nichtlinearen Parameterschätzung mittels des* Gauß-Jacobi Algorithmus. *Für einen Satz von dreidimensionalen Koordinaten, gegeben in einem Lokalsystem und im globalen WGS-Referenzsystem, werden die 7 Datum-Parameter bestimmt und zur Transformation weiterer Koordinatensätze verwendet, die nur im Lokalsystem vorliegen. Als Kriterium wird für beide algorithmische Lösungen das Maß der Anpassung bestimmt. Das Niveau der Anpassung ist für die nichtlineare Lösung etwas besser als für die linearisierte Lösung mittels der Methode der kleinsten Quadrate. Der vorgeschlagene* Gauß-Jacobi Algorithmus *zur nichtlinearen Parameterschätzung bietet folgende Vorteile: (i) Ausreißer werden* objektiv *erkannt, (ii) eine Linearisierung entfällt, (iii) Iterationen sind nicht notwendig, (iv) die Varianz-Kovarianz Matrizen beider Koordinatensätze (lokales System, globales Referenzsystem) werden automatisch berücksichtigt, (v) Ausreißer werden automatisch sichtbar gemacht.*

1 Introduction

The *7-parameter datum transformation* $\mathbb{C}_7(3)$ problem comprises the determination of seven parameters required to transform coordinates from one system to another. Transformation of coordinates is a computational procedure that maps one set of coordinates in a given system into another coordinate system. This is achieved by translating the given system so as to cater for its origin with respect to the final system and rotating about its own axes so as to orient it to the final system. In addition to translation and rotation, scaling is performed in order to match the corresponding baseline length in the two systems. The *three translation parameters*, *three rotation parameters* and the *scale element* comprise the *7-parameter datum transformation* $\mathbb{C}_7(3)$ *elements* that are necessary to transform a set of three-dimensional coordinates from one system into another system. In the *7-parameter datum transformation* $\mathbb{C}_7(3)$, one understands the $\mathbb{C}_7(3)$ to be the notion of the *seven parameter conformal group in* $\mathbb{R}^3$, leaving *space angles* and *distance ratios equivariant* (invariant). That the subject of transformation in general is still an active area of research is evidenced in the work of J. Guo and F. Jin (2001) who propose a new model of digitizing coordinate transformation. Recent communication between J. Reinking (2001), K.R. Koch (2001) and E. Lenzmann and L. Lenzmann (2001b) on the subject, following the work of E. Lenzmann and L. Lenzmann (2001a), provides an insight on the subject. In particular, the communication touches on the question of stochasticity of both systems involved in the solution of the *7-parameter datum transformation* problem.

The need for *7-parameter datum transformation* has been on the rise following the increase in positioning using GPS and GLONASS satellites both in Geodesy and Photogrammetry. The bottleneck to the problem however has been how to incorporate the stochasticity (variance-covariance) matrices of both systems involved. In practice, the users have been forced to rely on *iterative procedures* and *linearized Least Squares solution* which apart from the requirement of approximate starting values does not offer the possibility of incorporating the variance-covariance matrices of both systems in play. The desire to avoid the requirement of approximate starting values led to proposal on the use of the complete *Procrustes procedure* (e.g. E. Grafarend and J. Awange 2000). Literature on the *7-parameter datum transformation* problem include: B.R. Harvey (1986), O. Wolfrum (1992), P.A.M. Abusali et al. (1994), W.M. Welsch (1993), H. Abd-Elmotaal and M. El-Tokhey (1995), E. Grafarend et al. (1995), G. Birardi (1996), G. Kampmann (1996), P. Vanicek and R.R. Steeves (1996), M. Beranek (1997), W.E. Featherstone (1997), E. Grafarend and R. Syffus (1997,1998), F.I. Okeke (1998), E. Grafarend and F. Okeke (1998), J.L. Awange (1999), Y.A. Bazlov et al. (1999), Y. Yang (1999), Iz.H. Baki and Y.Q. Chen (1999), Koch K.R. et al. (2000), M. Nitschke and E.H. Knickmeyer (2000), E. Grafarend et al. (2000) and A. Barsi (2001).

In J.L. Awange and E. Grafarend (2002a), it was illustrated how the algebraic technique of *Groebner basis* solves explicitly the *nonlinear 7-parameter datum transformation equations* once they have been *converted* into algebraic (polynomial) form. In particular, the algebraic tool of *Groebner basis* provides symbolic solutions to the problem of *7-parameter datum transformation* and shows the *scale parameter* to fulfill a *quartic polynomial (univariate polynomial* of order *four)* while the rotation parameters are given by *linear functions* once the scale parameter has been determined (J.L. Awange 2002). Similar to the *Gauss elimination technique* used to solve linear systems of equations, the *Groebner basis* approach eliminates several variables in a multivariate system of nonlinear equations in such a manner that the end product (with the lexicographic ordering of monomial specified) normally consist of *univariate polynomial* equations whose roots can be determined by existing programs such as the *roots* command in MATLAB.

In J.L. Awange and E. Grafarend (2002b), we demonstrated how closed form solutions of the overdetermined *7-parameter datum transformation* problem could be achieved by forming the combinatorial solutions of the nine *7-parameter datum transformation* $\mathbb{C}_7(3)$ equations. In the present contribution, we apply the *linearized Least Squares solution* and the *Gauss-Jacobi combinatorial algorithm* to compute the *7-transformation parameters* and use the obtained values to transform coordinates from a local system to the WGS 84 system. We employ the *Groebner basis* in solving the minimum combinatorial subsets thereby using the *skew-symmetric* matrix to construct the orthogonal matrix $\boldsymbol{X}_3$. Other approaches for parameterizing the rotation matrix have been presented in G.H. Shut (1958/59) and E.H. Thompson (1959 a, b).

We organize the present study as follows; in Section 2 we present a brief outline of the procedures while Section 3 considers a case study. A summary is thereafter presented in Section 4.

2 Application of the procedures

The *7-parameters datum transformation* $\mathbb{C}_7(3)$ is given by the expression

$$\begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = x_1 \boldsymbol{X}_3 \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} + \boldsymbol{x}_2 \mid i = 1,2,3,\ldots,n \qquad (2\text{-}1)$$

subject to

$$\boxed{\boldsymbol{X}_3^T \boldsymbol{X}_3 = \boldsymbol{I}_3} \qquad (2\text{-}2)$$

with $\{a_i, b_i, c_i\}$ and $\{X_i, Y_i, Z_i\}$ being coordinates of the same points in both systems and $x_1 \in \mathbb{R}^3$, $x_2 \in \mathbb{R}^3$, $\boldsymbol{X}_3 \in \mathbb{R}^{3x3}$ are the unknowns. The task at hand is to obtain the seven unknown parameters of datum transformation $\mathbb{C}_7(3)$ defined by the scale factor $x_1 \in \mathbb{R}$, the translation elements in the vector $x_2 \in \mathbb{R}^3$, the rotation elements of the matrix of rotation $\boldsymbol{X}_3 \in \mathbb{R}^{3x3}$ and then use them to transform the coordinates from a given Local system to the WGS 84 system in order to obtain the corresponding transformed values. By making use of the *skew-symmetric* matrix $\boldsymbol{S}$, the rotation matrix is expressed as

$$\boldsymbol{X}_3 = (\boldsymbol{I}_3 - \boldsymbol{S})^{-1}(\boldsymbol{I}_3 + \boldsymbol{S}) \qquad (2\text{-}3)$$

where $\boldsymbol{I}_3$ is the identity matrix and the *skew-symmetric* matrix $\boldsymbol{S}$ given by

$$\boldsymbol{S} = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \qquad (2\text{-}4)$$

The rotation matrix $\boldsymbol{X}_3 \in \mathbb{R}^{3x3}$ is parameterized using either *Euler* or *Cardan* angles. Parameterization using Cardan angles is as presented in *Box* 2-1 on the following page.

Using *linearized Least Squares solution*, the rotation matrix (2-6) respectively (2-5) is inserted in (2-1) and the resulting expression linearized using Taylor series expansion about the approximate values of the unknowns. With the elements of the partial derivatives forming the design matrix and with coordinates of more than three stations known in both systems, the vector of unknown correction to the approximate values can be estimated using *linearized Least Squares solution*. The procedure can then be iterated until a given threshold is reached. The setback in using this approach as mentioned earlier is that given the variance-covariance matrices of the systems in operation (i.e. the local and the WGS 84 systems), it is difficult to incorporate them in the *linearized Least Squares approach*. The solution to this problem would be to turn to the *Gauss-Jacobi combinatorial algorithm* already proposed by J.L. Awange and E. Grafarend (2002b) and J.L. Awange (2002).

In J.L. Awange and E. Grafarend (2002a), a *quartic polynomial* (univariate polynomial of order four) for solving the scale parameters (*Box* 2-2) and the linear functions in terms of scale for solving the rotation parameters (*Box* 2-3) were derived using *Groebner basis*. This was made possible by parametrization of the rotation matrix via the *skew-symmetric* matrix (2-4).

Once the admissible value of scale parameter has been chosen amongst the four roots in *Box* (2-2) as $x_1 \in \mathbb{R}^+$, the

$$\boldsymbol{X}_3 = \boldsymbol{R}_1(\alpha)\boldsymbol{R}_2(\beta)\boldsymbol{R}_3(\gamma) \tag{2-5}$$

with

$$\boldsymbol{R}_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\alpha & sin\alpha \\ 0 & -sin\alpha & cos\alpha \end{bmatrix}, \boldsymbol{R}_2(\beta) = \begin{bmatrix} cos\beta & 0 & -sin\beta \\ 0 & 1 & 0 \\ sin\beta & 0 & cos\beta \end{bmatrix}, \boldsymbol{R}_3(\gamma) = \begin{bmatrix} cos\gamma & sin\gamma & 0 \\ -sin\gamma & cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

leading to

$$\boldsymbol{R}_1(\alpha)\boldsymbol{R}_2(\beta)\boldsymbol{R}_3(\gamma) = \begin{bmatrix} cos\beta cos\gamma & cos\beta sin\gamma & -sin\beta \\ sin\alpha sin\beta cos\gamma - cos\alpha sin\gamma & sin\alpha sin\beta sin\gamma + cos\alpha cos\gamma & sin\alpha cos\beta \\ cos\alpha sin\beta cos\gamma + sin\alpha sin\gamma & cos\alpha sin\beta sin\gamma - sin\alpha cos\gamma & cos\alpha cos\beta \end{bmatrix}. \tag{2-6}$$

The Cardan angles can be obtained from the rotation matrix $\boldsymbol{X}_3 \in \mathbb{R}^{3x3}$ through:

$$\begin{cases} \alpha = arctan\,(\frac{r_{23}}{r_{33}}) \\ \beta = arctan\,(\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{12}^2}}) \;\; or \;\; arctan\,(\frac{-r_{31}}{\sqrt{r_{23}^2 + r_{33}^2}}) \, . \\ \gamma = arctan\,(\frac{r_{12}}{r_{11}}) \end{cases} \tag{2-7}$$

Box 2-1: Parametrization of the rotation matrix by Cardan angles

$$\boxed{a_4 x_1^4 + a_3 x_1^3 + a_2 x_1^2 + a_1 x_1 + a_0 = 0}$$

$$\mathrm{a}_4 = (X_{13}Y_{12}^2Y_{23} + X_{12}^2X_{13}Y_{23} - X_{12}^2X_{23}Y_{13} - X_{12}Y_{13}Z_{12}Z_{23} - X_{23}Y_{12}^2Y_{13} + X_{13}Y_{12}Z_{12}Z_{23} - X_{23}Y_{12}Z_{12}Z_{13} + X_{12}Y_{23}Z_{12}Z_{13})$$

$$\mathrm{a}_3 = (c_{12}X_{13}Y_{12}Z_{23} - b_{13}X_{12}Z_{12}Z_{23} - c_{13}X_{12}Y_{23}Z_{12} + b_{23}X_{13}Y_{12}^2 - a_{23}X_{12}^2Y_{13} + c_{23}X_{12}Y_{13}Z_{12} + c_{12}X_{12}Y_{23}Z_{13} + a_{13}Y_{12}^2Y_{23} - b_{12}X_{23}Z_{12}Z_{13} - a_{23}Y_{12}^2Y_{13} + b_{12}X_{13}Z_{12}Z_{23} - c_{12}X_{12}Y_{13}Z_{23} + a_{13}X_{12}^2Y_{23} - c_{23}X_{13}Y_{12}Z_{12} - a_{23}Y_{12}Z_{12}Z_{13} + c_{13}X_{23}Y_{12}Z_{12} - b_{13}X_{23}Y_{12}^2 - b_{13}X_{12}^2X_{23} + b_{23}X_{12}Z_{12}Z_{13} + a_{13}Y_{12}Z_{12}Z_{23} + b_{23}X_{12}^2X_{13} + a_{12}Y_{23}Z_{12}Z_{13} - a_{12}Y_{13}Z_{12}Z_{23} - c_{12}X_{23}Y_{12}Z_{13})$$

$$\mathrm{a}_2 = (a_{13}b_{23}X_{12}^2 + b_{12}^2X_{23}Y_{13} + b_{12}c_{13}X_{23}Z_{12} - c_{12}c_{23}X_{13}Y_{12} + b_{13}c_{23}X_{12}Z_{12} - a_{23}b_{12}Z_{12}Z_{13} + a_{12}^2X_{23}Y_{13} - b_{12}^2X_{13}Y_{23} - a_{12}^2X_{13}Y_{23} - a_{23}b_{13}X_{12}^2 + a_{13}b_{23}Y_{12}^2 - a_{23}b_{13}Y_{12}^2 + a_{12}b_{23}Z_{12}Z_{13} + a_{23}c_{13}Y_{12}Z_{12} + a_{12}c_{12}Y_{23}Z_{13} - b_{12}c_{12}X_{23}Z_{13} - b_{23}c_{13}X_{12}Z_{12} - a_{12}b_{13}Z_{12}Z_{23} - a_{23}c_{12}Y_{12}Z_{13} + a_{12}c_{23}Y_{13}Z_{12} + b_{12}c_{12}X_{13}Z_{23} - a_{12}c_{12}Y_{13}Z_{23} - a_{13}c_{23}Y_{12}Z_{12} - c_{12}c_{13}X_{12}Y_{23} + c_{12}c_{13}X_{23}Y_{12} + c_{12}c_{23}X_{12}Y_{13} - b_{13}c_{12}X_{12}Z_{23} + b_{23}c_{12}X_{12}Z_{13} + a_{13}b_{12}Z_{12}Z_{23} - a_{12}c_{13}Y_{23}Z_{12} + a_{13}c_{12}Y_{12}Z_{23} - b_{12}c_{23}X_{13}Z_{12})$$

$$\mathrm{a}_1 = (-a_{12}b_{13}c_{12}Z_{23} + b_{12}c_{12}c_{13}X_{23} + b_{12}^2b_{13}X_{23} - a_{13}b_{12}^2Y_{23} - a_{12}^2b_{23}X_{13} + a_{23}c_{12}c_{13}Y_{12} - a_{13}c_{12}c_{23}Y_{12} + a_{12}b_{13}c_{23}Z_{12} - a_{12}^2a_{13}Y_{23} - b_{23}c_{12}c_{13}X_{12} + a_{12}b_{23}c_{12}Z_{13} - a_{23}b_{12}c_{12}Z_{13} - b_{12}^2b_{23}X_{13} + a_{23}b_{12}c_{13}Z_{12} - a_{12}c_{12}c_{13}Y_{23} + a_{12}^2b_{13}X_{23} + a_{12}^2a_{23}Y_{13} - a_{12}b_{23}c_{13}Z_{12} - a_{13}b_{12}c_{23}Z_{12} - b_{12}c_{12}c_{23}X_{13} + b_{13}c_{12}c_{23}X_{12} + a_{12}c_{12}c_{23}Y_{13} + a_{23}b_{12}^2Y_{13} + a_{13}b_{12}c_{12}Z_{23})$$

$$\mathrm{a}_0 = a_{12}b_{13}c_{12}c_{23} - a_{13}b_{12}^2b_{23} + a_{12}^2a_{23}b_{13} - a_{12}^2a_{13}b_{23} - a_{12}b_{23}c_{12}c_{13} + a_{23}b_{12}^2b_{13} + a_{23}b_{12}c_{12}c_{13} - a_{13}b_{12}c_{12}c_{23}$$

Box 2-2: Quartic polynomial for computing scale parameter

$$f(a)=(-x_1^3X_{13}Y_{12}Z_{12} - b_{12}x_1^2X_{13}Z_{12} - a_{13}b_{12}c_{12} - a_{13}x_1^2Y_{12}Z_{12} + b_{13}x_1^2X_{12}Z_{12} + c_{12}x_1^2X_{12}Y_{13} - a_{13}c_{12}x_1Y_{12} + a_{12}b_{13}c_{12} + a_{12}b_{13}x_1Z_{12} - a_{13}b_{12}x_1Z_{12} - b_{12}c_{12}x_1X_{13} + a_{12}x_1^2Y_{13}Z_{12} + x_1^3X_{12}Y_{13}Z_{12} + a_{12}c_{12}x_1Y_{13} - c_{12}x_1^2X_{13}Y_{12} + b_{13}c_{12}x_1X_{12})a + (-a_{12}^2a_{13} - a_{12}c_{13}x_1Z_{12} + a_{12}x_1^2Z_{12}Z_{13} + a_{12}c_{12}x_1Z_{13} - c_{12}c_{13}x_1X_{12} - a_{12}c_{12}c_{13} + x_1^3X_{13}Y_{12}^2 - c_{13}x_1^2X_{12}Z_{12} + x_1^3X_{12}^2X_{13} + x_1^3X_{12}Z_{12}Z_{13} - a_{12}^2x_1X_{13} - b_{12}^2x_1X_{13} - a_{13}b_{12}^2 + a_{13}x_1^2X_{12}^2 + a_{13}x_1^2Y_{12}^2 + c_{12}x_1^2X_{12}Z_{13})$$

$$f(b)=(-b_{12}c_{12}x_1X_{13} + b_{13}c_{12}x_1X_{12} - a_{13}b_{12}x_1Z_{12} - x_1^3X_{13}Y_{12}Z_{12} - a_{13}c_{12}x_1Y_{12} + a_{12}b_{13}c_{12} + a_{12}x_1^2Y_{13}Z_{12} - a_{13}x_1^2Y_{12}Z_{12} + a_{12}b_{13}x_1Z_{12} + x_1^3X_{12}Y_{13}Z_{12} - a_{13}b_{12}c_{12} + c_{12}x_1^2X_{12}Y_{13} + a_{12}c_{12}x_1Y_{13} - b_{12}x_1^2X_{13}Z_{12} - c_{12}x_1^2X_{13}Y_{12} + b_{13}x_1^2X_{12}Z_{12})b + (-a_{12}^2b_{13} - c_{12}c_{13}x_1Y_{12} + b_{13}x_1^2Y_{12}^2 + c_{12}x_1^2Y_{12}Z_{13} + x_1^3Y_{12}Z_{12}Z_{13} + x_1^3X_{12}^2Y_{13} - b_{12}c_{13}x_1Z_{12} + b_{12}x_1^2Z_{12}Z_{13} - b_{12}^2b_{13} - a_{12}^2x_1Y_{13} + b_{13}x_1^2X_{12}^2 - c_{13}x_1^2Y_{12}Z_{12} + b_{12}c_{12}x_1Z_{13} + x_1^3Y_{12}^2Y_{13} - b_{12}^2x_1Y_{13} - b_{12}c_{12}c_{13})$$

$$f(c)=(-b_{12}x_1X_{13} - a_{13}x_1Y_{12} - x_1^2X_{13}Y_{12} - a_{13}b_{12} + x_1^2X_{12}Y_{13} + a_{12}b_{13} + a_{12}x_1Y_{13} + b_{13}x_1X_{12})c + (-a_{12}a_{13} - b_{12}b_{13} + x_1^2Z_{12}Z_{13} + x_1^2Y_{12}Y_{13} - c_{12}c_{13} + a_{13}x_1X_{12} + b_{13}x_1Y_{12} - a_{12}x_1X_{13} + x_1^2X_{12}X_{13} + c_{12}x_1Z_{13} - b_{12}x_1Y_{13} - c_{13}x_1Z_{12}).$$

Box 2-3: Linear functions for computing the parameters of the skew-symmetric matrix S

elements of the *skew-symmetric* matrix $\boldsymbol{S}$ can then be obtained via the linear functions in scale in *Box* (2-3).

From the elements of the *skew-symmetric* matrix $\boldsymbol{S}$ in *Box* (2-3), the rotation matrix $\boldsymbol{X}_3$ from which the Cardan rotation angles are deduced is formed. The translation elements $\boldsymbol{x}_2$ can then be computed by substituting the scale parameter x_1 and the rotation matrix $\boldsymbol{X}_3$ in (2-1).

Given n stations in both systems and with the minimum number of stations required for a closed form solution of the unknowns being 3, one forms

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)\dots(n-3+1)}{3 \times \dots 2 \times 1} \quad (2\text{-}8)$$

minimal combinatorial subsets, each solved using the algebraic expressions of *Boxes* (2-2) and (2-3) to give the values of scale and rotation elements. With the computation of these minimum combinatorial subsets goes the computation of their respective variance-covariance matrices from the *nonlinear error propagation law/variance-covariance propagation law*. From a combinatorial subset comprising three points in both system involved in the transformation process, 9 nonlinear equations

$$\begin{bmatrix} f_1 := x_1X_1 - x_1cY_1 + x_1bZ_1 + X_0 - a_1 - cb_1 + bc_1 = 0 \\ f_2 := x_1cX_1 + x_1Y_1 - x_1aZ_1 + Y_0 + ca_1 - b_1 - ac_1 = 0 \\ f_3 := -x_1bX_1 + x_1aY_1 + x_1Z_1 + Z_0 - ba_1 + ab_1 - c_1 = 0 \\ f_4 := x_1X_2 - x_1cY_2 + x_1bZ_2 + X_0 - a_2 - cb_2 + bc_2 = 0 \\ f_5 := x_1cX_2 + x_1Y_2 - x_1aZ_2 + Y_0 + ca_2 - b_2 - ac_2 = 0 \\ f_6 := -x_1bX_2 + x_1aY_2 + x_1Z_2 + Z_0 - ba_2 + ab_2 - c_2 = 0 \\ f_7 := x_1X_3 - x_1cY_3 + x_1bZ_3 + X_0 - a_3 - cb_3 + bc_3 = 0 \\ f_8 := x_1cX_3 + x_1Y_3 - x_1aZ_3 + Y_0 + ca_3 - b_3 - ac_3 = 0 \\ f_9 := -x_1bX_3 + x_1aY_3 + x_1Z_3 + Z_0 - ba_3 + ab_3 - c_3 = 0 \end{bmatrix} \quad (2\text{-}9)$$

are formed with $\{x_i, y_i, z_i\} := \{a_i, b_i, c_i\} \mid i \in \{1, 2, 3\}$, equations $\{f_1, f_2, f_3\}$ being equations formed from the *first point* with coordinates in both systems, $\{f_4, f_5, f_6\}$ being equations formed from the *second point* with coordinates in both systems and $\{f_7, f_8, f_9\}$ being the equations formed from the *third point* with coordinates in both systems. From (2-9), one requires only seven equations to solve for the *7-parameter datum transformation* problem (J.L. Awange and E. Grafarend 2002a). Let us consider the system of nonlinear equations extracted from (2-9) to be formed by $\{f_1, f_2, f_3, f_4, f_5, f_6, f_9\}$ and solved in closed form using *Groebner basis* approach to give symbolic solutions in the form given by *Boxes* (2-2) and (2-3). For n points with coordinates in both systems, we form combinatorials via (2-8) and solve for scale and rotation parameters using the algorithms of *Boxes* (2-2) and (2-3). The combinatorial solution from each subset are then used in the *nonlinear error propagation law/variance-covariance propagation law*. From the algebraic system of equations formed by $\{f_1, f_2, f_3, f_4, f_5, f_6, f_9\}$ from (2-9), we have the Jacobi matrices as

$$\boldsymbol{J}_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b} & \frac{\partial f_1}{\partial c} & \frac{\partial f_1}{\partial X_0} & \frac{\partial f_1}{\partial Y_0} & \frac{\partial f_1}{\partial Z_0} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} & \frac{\partial f_2}{\partial c} & \frac{\partial f_2}{\partial X_0} & \frac{\partial f_2}{\partial Y_0} & \frac{\partial f_2}{\partial Z_0} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial a} & \frac{\partial f_3}{\partial b} & \frac{\partial f_3}{\partial c} & \frac{\partial f_3}{\partial X_0} & \frac{\partial f_3}{\partial Y_0} & \frac{\partial f_3}{\partial Z_0} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial a} & \frac{\partial f_4}{\partial b} & \frac{\partial f_4}{\partial c} & \frac{\partial f_4}{\partial X_0} & \frac{\partial f_4}{\partial Y_0} & \frac{\partial f_4}{\partial Z_0} \\ \frac{\partial f_5}{\partial x_1} & \frac{\partial f_5}{\partial a} & \frac{\partial f_5}{\partial b} & \frac{\partial f_5}{\partial c} & \frac{\partial f_5}{\partial X_0} & \frac{\partial f_5}{\partial Y_0} & \frac{\partial f_5}{\partial Z_0} \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial a} & \frac{\partial f_6}{\partial b} & \frac{\partial f_6}{\partial c} & \frac{\partial f_6}{\partial X_0} & \frac{\partial f_6}{\partial Y_0} & \frac{\partial f_6}{\partial Z_0} \\ \frac{\partial f_9}{\partial x_1} & \frac{\partial f_9}{\partial a} & \frac{\partial f_9}{\partial b} & \frac{\partial f_9}{\partial c} & \frac{\partial f_9}{\partial X_0} & \frac{\partial f_9}{\partial Y_0} & \frac{\partial f_9}{\partial Z_0} \end{bmatrix} \quad (2\text{-}10)$$

and

$$J_y = \begin{bmatrix} \frac{\partial f_1}{\partial a_1} & \frac{\partial f_1}{\partial b_1} & \frac{\partial f_1}{\partial c_1} & \frac{\partial f_1}{\partial a_2} & \frac{\partial f_1}{\partial b_2} & \frac{\partial f_1}{\partial c_2} & \frac{\partial f_1}{\partial a_3} & \frac{\partial f_1}{\partial b_3} & \frac{\partial f_1}{\partial c_3} & \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial Y_1} & \frac{\partial f_1}{\partial Z_1} & \cdots & \frac{\partial f_1}{\partial X_3} & \frac{\partial f_1}{\partial Y_3} & \frac{\partial f_1}{\partial Z_3} \\ \frac{\partial f_2}{\partial a_1} & \frac{\partial f_2}{\partial b_1} & \frac{\partial f_2}{\partial c_1} & \frac{\partial f_2}{\partial a_2} & \frac{\partial f_2}{\partial b_2} & \frac{\partial f_2}{\partial c_2} & \frac{\partial f_2}{\partial a_3} & \frac{\partial f_2}{\partial b_3} & \frac{\partial f_2}{\partial c_3} & \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial Y_1} & \frac{\partial f_2}{\partial Z_1} & \cdots & \frac{\partial f_2}{\partial X_3} & \frac{\partial f_2}{\partial Y_3} & \frac{\partial f_2}{\partial Z_3} \\ \frac{\partial f_3}{\partial a_1} & \frac{\partial f_3}{\partial b_1} & \frac{\partial f_3}{\partial c_1} & \frac{\partial f_3}{\partial a_2} & \frac{\partial f_3}{\partial b_2} & \frac{\partial f_3}{\partial c_2} & \frac{\partial f_3}{\partial a_3} & \frac{\partial f_3}{\partial b_3} & \frac{\partial f_3}{\partial c_3} & \frac{\partial f_3}{\partial X_1} & \frac{\partial f_3}{\partial Y_1} & \frac{\partial f_3}{\partial Z_1} & \cdots & \frac{\partial f_3}{\partial X_3} & \frac{\partial f_3}{\partial Y_3} & \frac{\partial f_3}{\partial Z_3} \\ \frac{\partial f_4}{\partial a_1} & \frac{\partial f_4}{\partial b_1} & \frac{\partial f_4}{\partial c_1} & \frac{\partial f_4}{\partial a_2} & \frac{\partial f_4}{\partial b_2} & \frac{\partial f_4}{\partial c_2} & \frac{\partial f_4}{\partial a_3} & \frac{\partial f_4}{\partial b_3} & \frac{\partial f_4}{\partial c_3} & \frac{\partial f_4}{\partial X_1} & \frac{\partial f_4}{\partial Y_1} & \frac{\partial f_4}{\partial Z_1} & \cdots & \frac{\partial f_4}{\partial X_3} & \frac{\partial f_4}{\partial Y_3} & \frac{\partial f_4}{\partial Z_3} \\ \frac{\partial f_5}{\partial a_1} & \frac{\partial f_5}{\partial b_1} & \frac{\partial f_5}{\partial c_1} & \frac{\partial f_5}{\partial a_2} & \frac{\partial f_5}{\partial b_2} & \frac{\partial f_5}{\partial c_2} & \frac{\partial f_5}{\partial a_3} & \frac{\partial f_5}{\partial b_3} & \frac{\partial f_5}{\partial c_3} & \frac{\partial f_5}{\partial X_1} & \frac{\partial f_5}{\partial Y_1} & \frac{\partial f_5}{\partial Z_1} & \cdots & \frac{\partial f_5}{\partial X_3} & \frac{\partial f_5}{\partial Y_3} & \frac{\partial f_5}{\partial Z_3} \\ \frac{\partial f_6}{\partial a_1} & \frac{\partial f_6}{\partial b_1} & \frac{\partial f_6}{\partial c_1} & \frac{\partial f_6}{\partial a_2} & \frac{\partial f_6}{\partial b_2} & \frac{\partial f_6}{\partial c_2} & \frac{\partial f_6}{\partial a_3} & \frac{\partial f_6}{\partial b_3} & \frac{\partial f_6}{\partial c_3} & \frac{\partial f_6}{\partial X_1} & \frac{\partial f_6}{\partial Y_1} & \frac{\partial f_6}{\partial Z_1} & \cdots & \frac{\partial f_6}{\partial X_3} & \frac{\partial f_6}{\partial Y_3} & \frac{\partial f_6}{\partial Z_3} \\ \frac{\partial f_9}{\partial a_1} & \frac{\partial f_9}{\partial b_1} & \frac{\partial f_9}{\partial c_1} & \frac{\partial f_9}{\partial a_2} & \frac{\partial f_9}{\partial b_2} & \frac{\partial f_9}{\partial c_2} & \frac{\partial f_9}{\partial a_3} & \frac{\partial f_9}{\partial b_3} & \frac{\partial f_9}{\partial c_3} & \frac{\partial f_9}{\partial X_1} & \frac{\partial f_9}{\partial Y_1} & \frac{\partial f_9}{\partial Z_1} & \cdots & \frac{\partial f_9}{\partial X_3} & \frac{\partial f_9}{\partial Y_3} & \frac{\partial f_9}{\partial Z_3} \end{bmatrix} \quad (2\text{-}11)$$

where the dotted points in J_y represent the partial derivative of $\{f_1, f_2, f_3, f_4, f_5, f_6, f_9\}$ formed from (2-9) with respect $\{X_2, Y_2, Z_2\}$. From the dispersion Σ_y of the vector of observations y and with $J = J_x^{-1} J_y$, the dispersion matrix Σ_x is then obtained from $\Sigma_x = J \Sigma_y J'$, while the correlation between the ith and the jth combination is given by $\Sigma_{ij} = J_j \Sigma_{yjyi} J_j'$. These variance-covariance matrices and the correlations are then joined and used in the sparse matrix technique and used in the *linearized Gauss-Markov* model to obtain the weighted mean of the combinatorial solutions as discussed in J.L. Awange and E. Grafarend (2002b) and J.L. Awange (2002).

3 Case study

We consider Cartesian coordinates of seven stations given in the Local and Global Reference Systems (WGS 84) as in *Tables* (1) and (2). Desired are the seven parameters of datum transformation. First we computed the 7-transformation parameters using the *linearized Least Squares solution* and used the computed values to transform the coordinates from the Local system (*Table* 1) to the WGS 84 system (*Table* 2). Next, the computation is repeated using the *Gauss-Jacobi combinatorial* solution. From the values of *Tables* (1) and (2), 36 minimum combinatorial solutions are formed from (2-8) and solved in closed form using *Groebner basis* approach (J.L. Awange and E. Grafarend 2002a). The resulting combinatorial solutions are then adjusted to their barycenter by making use of the obtained variance-covariance matrix from *nonlinear error propagation law/variance-covariance propagation law*.

Transformation parameters obtained by the *Gauss-Jacobi combinatorial algorithm* are used to transform the Cartesian coordinates from the *Local Reference System* (*Table* 1) to the *Global Reference System* (WGS 84, *Table* 2) as shown in *Table* (3). *Table* (4) gives for comparison purposes the transformed values using the 7-datum transformation parameters obtained via *linearized Least Squares solution*. The residuals from both *Gauss-Jacobi combinatorial algorithm* and *linearized Least Squares solution* are in the same range in magnitude. We also compute the residual norm (square root of the sum of squares of residuals) and present them in *Table* (5). *Figure* (1) shows the scatter of the computed 36 minimal combinatorial solutions of scale (indicated by dotted points (•)) around for the adjusted value indicated by a line (–). *Figures* (2) and (3) show the scatter of the computed 36 minimal combinatorial solutions of translation and rotation parameters (indicated by dotted points (•)) around the adjusted values indicated by a star (*). The Figures clearly identify the outlying combinations from which the respective (suspected outlying points) points can be deduced.

Station Name	*X (m)*	*Y (m)*	*Z (m)*
Solitude	4157222.543	664789.307	4774952.099
Buoch Zeil	4149043.336	688836.443	4778632.188
Hohenneuffen	4172803.511	690340.078	4758129.701
Kuehlenberg	4177148.376	642997.635	4760764.800
Ex Mergelaec	4137012.190	671808.029	4791128.215
Ex Hof Asperg	4146292.729	666952.887	4783859.856
Ex Kaisersbach	4138759.902	702670.738	4785552.196

Table 1: Coordinates for system A (Local system)

Station Name	*X (m)*	*Y (m)*	*Z (m)*
Solitude	4157870.237	664818.678	4775416.524
Buoch Zeil	4149691.049	688865.785	4779096.588
Hohenneuffen	4173451.354	690369.375	4758594.075
Kuehlenberg	4177796.064	643026.700	4761228.899
Ex Mergelaec	4137659.549	671837.337	4791592.531
Ex Hof Asperg	4146940.228	666982.151	4784324.099
Ex Kaisersbach	4139407.506	702700.227	4786016.645

Table 2: Coordinates for system B (WGS 84)

Site	X (m)	Y (m)	Z (m)
System A: Solitude	4157222.5430	664789.3070	4774952.0990
System B	4157870.2370	664818.6780	4775416.5240
Transformed value	4157870.1631	664818.5399	4775416.3843
Residual	0.0739	0.1381	0.1397
System A: Buoch Zeil	4149043.3360	688836.4430	4778632.1880
System B	4149691.0490	688865.7850	4779096.5880
Transformed value	4149691.0162	688865.8151	4779096.5785
Residual	0.0328	-0.0301	0.0095
System A: Hohenneuffen	4172803.5110	690340.0780	4758129.7010
System B	4173451.3540	690369.3750	4758594.0750
Transformed value	4173451.3837	690369.4437	4758594.0770
Residual	-0.0297	-0.0687	-0.0020
System A: Kuelenberg	4177148.3760	642997.6350	4760764.8000
System B	4177796.0640	643026.7000	4761228.8990
Transformed value	4177796.0394	643026.7347	4761228.9783
Residual	0.0246	-0.0347	-0.0793
System A: Ex Mergelaec	4137012.1900	671808.0290	4791128.2150
System B	4137659.5490	671837.3370	4791592.5310
Transformed value	4137659.6895	671837.3142	4791592.5458
Residual	-0.1405	0.0228	-0.0148
System A: Ex Hof Asperg	4146292.7290	666952.8870	4783859.8560
System B	4146940.2280	666982.1510	4784324.0990
Transformed value	4146940.2757	666982.1394	4784324.1589
Residual	-0.0477	0.0116	-0.0599
System A: Ex Keisersbach	4138759.9020	702670.7380	4785552.1960
System B	4139407.5060	702700.2270	4786016.6450
Transformed value	4139407.5733	702700.1935	4786016.6520
Residual	-0.0673	0.0335	-0.0070

Table 3: Transformed Cartesian coordinates of system A (Table 1) into System B (Table 2) using the 7-datum transformation parameters obtained from the Gauss-Jacobi combinatorial algorithm

Site	X (m)	Y (m)	Z (m)
System A: Solitude	4157222.5430	664789.3070	4774952.0990
System B	4157870.2370	664818.6780	4775416.5240
Transformed value	4157870.1430	664818.5429	4775416.3838
Residual	0.0940	0.1351	0.1402
System A: Buoch Zeil	4149043.3360	688836.4430	4778632.1880
System B	4149691.0490	688865.7850	4779096.5880
Transformed value	4149690.9902	688865.8347	4779096.5743
Residual	0.0588	-0.0497	0.0137
System A: Hohenneuffen	4172803.5110	690340.0780	4758129.7010
System B	4173451.3540	690369.3750	4758594.0750
Transformed value	4173451.3939	690369.4629	4758594.0831
Residual	-0.0399	-0.0879	-0.0081
System A: Kuelenberg	4177148.3760	642997.6350	4760764.8000
System B	4177796.0640	643026.7000	4761228.8990
Transformed value	4177796.0438	643026.7220	4761228.9864
Residual	0.0202	-0.0220	-0.0874
System A: Ex Mergelaec	4137012.1900	671808.0290	4791128.2150
System B	4137659.5490	671837.3370	4791592.5310
Transformed value	4137659.6409	671837.3231	4791592.5365
Residual	-0.0919	0.0139	-0.0055
System A: Ex Hof Asperg	4146292.7290	666952.8870	4783859.8560
System B	4146940.2280	666982.1510	4784324.0990
Transformed value	4146940.2398	666982.1445	4784324.1536
Residual	-0.0118	0.0065	-0.0546
System A: Ex Keisersbach	4138759.9020	702670.7380	4785552.1960
System B	4139407.5060	702700.2270	4786016.6450
Transformed value	4139407.5354	702700.2229	4786016.6433
Residual	-0.0294	0.0041	-0.0017

Table 4: Transformed Cartesian coordinates of system A (Table 1) into System B (Table 2) using the 7-datum transformation parameters computed by »linearized Least Squares solution«

Method	X (m)	Y (m)	Z (m)
Linearized Least Squares Solution	0.1541	0.1708	0.1748
Nonlinear Gauss-Jacobi Combinatorial	0.1859	0.1664	0.1725

Table 5: Computed residual norms

4 Summary

In the present example, the results obtained by both procedures give residuals that are in the same range in magnitude. We have demonstrated that with proper combinatorial subsets formed and given the stochasticity of both systems, the Gauss-Jacobi combinatorial algorithm could offer a remedy. Besides, it is suitable for outlier identification as opposed to the *linearized Least Squares solution*. These can be seen from the scatter *Figures* (1) to (3).

References

Abd-Elmotaal, H., El-Tokhey, M.: Effect of spherical approximation on datum transformation. Manuscripta Geodaetica 20: 469–474, 1995.

Abusali, P.A.M., Schutz, B.E., Tapley, B.D.: Transformation between SLR/VLBI and WGS-84 frames. Bulletin Geodesique 69: 61–72, 1994.

Awange, J.L.: Partial procrustes solution of the threedimensional orientation problem from GPS/LPS observations. In: F. Krumm and V.S. Schwarze (eds.). Quo vadis geodesia …? Festschrift to E.W. Grafarend on the occasion of his 60th birthday, Report Nr. 1999.6–1, 1999.

Awange, J.L.: Gröbner basis, multipolynomial resultants and the Gauss-Jacobi combinatorial algorithms – adjustment of nonlinear

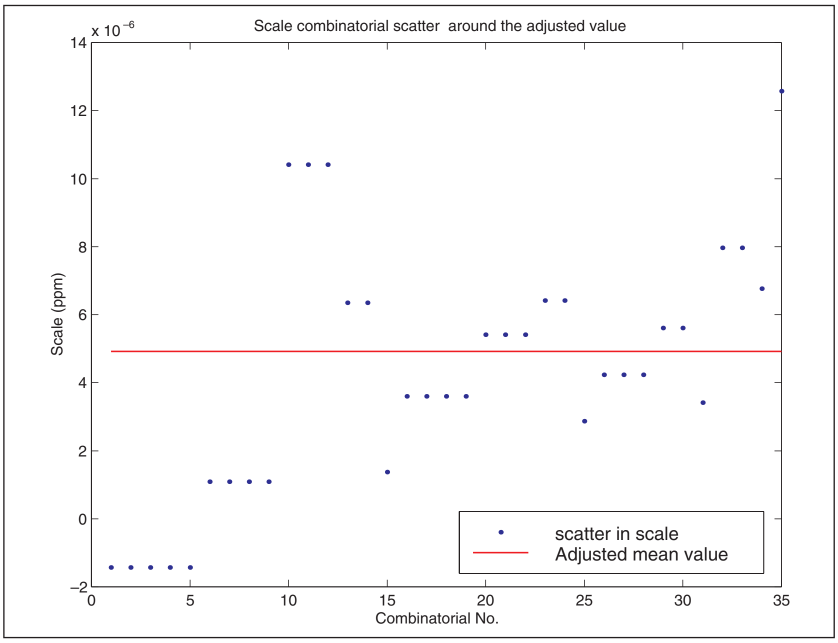

Fig. 1: Scatter of the computed 36 minimal combinatorial values of scale around the adjusted value

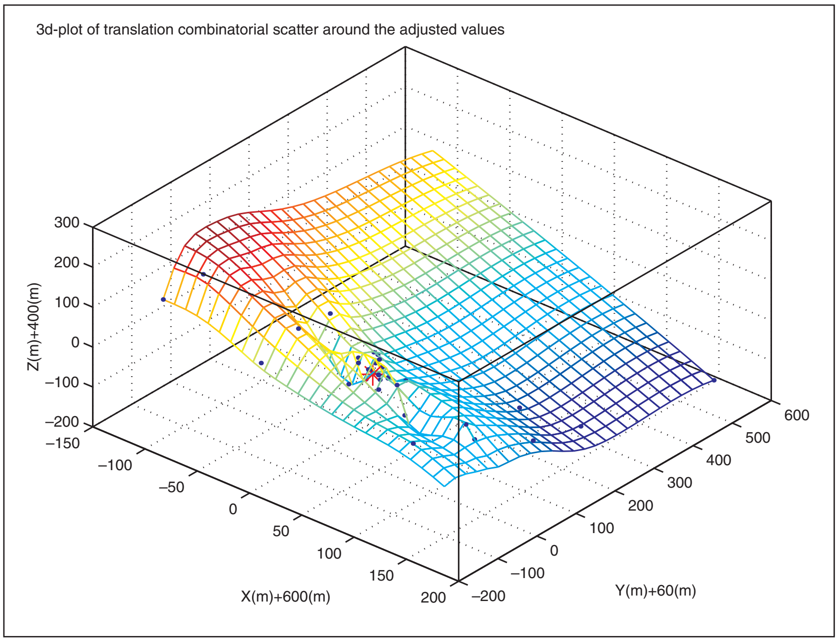

Fig. 2: Scatter of the 36 computed translations about the adjusted values

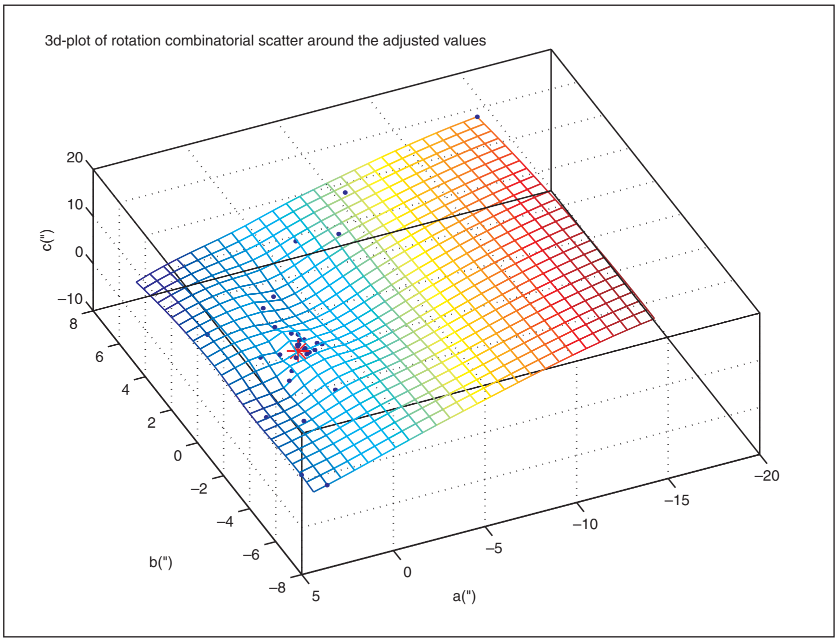

Fig. 3: Scatter of the 36 computed rotations about the adjusted values

GPS/LPS observations. Dissertation. Technical Reports, Department of Geodesy and GeoInformatics, Report Nr. 2002 (1).

Awange, J. L., Grafarend, E.: Closed form solution of the nonlinear 7 parameter datum transformation by Groebner basis. Under review in the Journal of Geodesy, 2002a.

Awange, J. L., Grafarend, E.: Closed form solution of the overdetermined nonlinear 7 parameter datum transformation. Under review in the Journal of Geodesy, 2002b.

Baki, Iz. H., Chen, Y. Q.: Tailored datum transformation model for locally distributed data. Journal of Surveying Engineering 125: 25–35, 1999.

Barsi, A.: Performing coordinate transformation by artificial neural network. Allgemeine VermessungsNachrichten 108: 134–137, 2001.

Bazlov, Y. A., Galazin, V. F., Kaplan, B. L., Maksimov, V. G., Rogozin, V. P.: Propagating PZ 90 and WGS 94 transformation parameters. GPS Solutions 3(1): 13–16, 1999.

Beranek, M.: Überlegungen zur Normalverteilung und theoretische Analyse der 3- und 4-Parameter-Transformation. Allgemeine Vermessungs-Nachrichten 104: 137–141, 1997.

Birardi, G.: The future global geodetic network and the transformation of local onto WGS coordinates. Bullettino di Geodesia e Scienze Affini 55: 49–56, 1996.

Featherstone, W. E.: A comparison of existing co-orninate transformation models and parameters in Australia. Cartography 26: 13–25, 1997.

Grafarend, E., Syffus, R.: Strip transformation of conformal coordinates of type Gauss-Kruger and UTM. Allgemeine Vermessungs-Nachrichten 104: 184–190, 1997.

Grafarend, E., Syffus, R.: Transformation of conformal coordinates of type Gauss-Krüger or UTM from local datum (regional, national, European) to global datum (WGS 84) part 1: The transformation equations. Allgemeine Vermessungs-Nachrichten 105: 134–141, 1998.

Grafarend, E., Okeke, F.: Transformation of conformal coordinates of type Mercator from global datum (WGS 84) to local datum (regional, national). Marine Geodesy 21: 169–180, 1998.

Grafarend, E., Awange, J. L.: Determination of vertical deflections by GPS/LPS measurements. zfv 125: 279–288, 2000.

Grafarend, E., Krumm, F., Okeke, F.: Curvilinear geodetic datum transformation. zfv 120: 334–350, 1995.

Grafarend, E., Hendricks, A., Gilbert, A.: Transformation of conformal coordinates of type Gauss-Kruger or UTM from a local datum (Regional, National, European) to a global datum (WGS 84, ITRF 96). Allgemeine Vermessungs-Nachrichten 107: 218–222, 2000.

Guo, J., Jin, F.: A new model of digitizing coordinate transformation and its nonlinear solution. Allgemeine Vermessungs-Nachrichten 108: 311–317, 2001.

Harvey, B. R.: Transformation of 3D coordinates. The Australian Surveyor 33: 105–125, 1986.

Kampmann, G.: New adjustment techniques for the determination of transformation parameters for Cadastral and Engineering purposes. Geomatica 50: 27–34, 1996.

Koch K. R., Fröhlich, H., Bröker, G.: Transformation räumlicher variabler Koordinaten. Allgemeine Vermessungs-Nachrichten 107: 293–295, 2000.

Koch K. R.: Bemerkung zu der Veröffentlichung »Zur Bestimmung eindeutiger Transformationparameter«. zfv 126: 297, 2001.

Lenzmann, E., Lenzmann, L.: Zur Bestimmung eindeutiger Transformationparameter. zfv 126: 138–142, 2001a.

Lenzmann, E., Lenzmann, L.: Erwiderung auf die Anmerkung von Jörg Reinking und die Bemerkungen von Karl-Rudolf Koch zu unserem Beitrag »Zur Bestimmung eindeutiger Transformationparameter«. zfv 126: 298–299, 2001b.

Nitschke, M., Knickmeyer, E. H.: Rotation parameters – a survey of techniques. Journal of Surveying Engineering 126: 83–105, 2000.

Okeke, F. I.: The curvilinear datum transformation model. DGK, Reihe C, Heft Nr. 481, 1998.

Reinking, J.: Anmerkung zu »Zur Bestimmung eindeutiger Transformationparameter«. zfv 126: 295–296, 2001.

Shut, G. H.: Construction of orthogonal matrices and their application in analytical Photogrammetrie. Photogrammetria XV: 149–162, 1958/59.

Thompson, E. H.: A method for the construction of orthogonal matrices. Photogrammetria III: 55–59, 1959a.

Thompson, E. H.: An exact linear solution of the absolute orientation. Photogrammetria XV: 163–179, 1959b.

Vanicek, P., Steeves, R. R.: Transformation of coordinates between two horizontal geodetic datums. Journal of Geodesy 70: 740–745, 1996.

Welsch, W. M.: A general 7-parameter transformation for the combination, comparison and accuracy control of the terrestrial and satellite network observations. Manuscripta Geodaetica 18: 295–305, 1993.

Wolfrum, O.: Merging terestrial and satellite networks by a ten-parameter transformation model. Manuscripta Geodaetica 17: 210–214, 1992.

Yang Y.: Robust estimation of geodetic datum transformation. Journal of Geodesy 73: 268–274, 1999.

Authors' address

Dr.-Ing. Joseph L. Awange
Prof. Dr.-Ing. Dr. h. c. mult. Erik W. Grafarend
Department of Geodesy and GeoInformatics, University of Stuttgart
Geschwister-Scholl-Str. 24D, 70174 Stuttgart, Germany.
Tel.:+49-711-1213474, +49-711-1213389, Fax:+49-711-1213285
E-mail: awange@gis.uni-stuttgart.de, grafarend@gis.uni-stuttgart.de