

# A Numerical Method for Determining the Spatial HELMERT Transformation in the Case of Different Scale Factors

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## Summary

The HELMERT transformation is needed and used in geodesy. One tries to find a special affine transformation, consisting of a translation and a rotation multiplied by a scale factor so that one given set of points in 3-d space is approximately transformed into a second one. We extend the model to the case of different scale factors for each of the three coordinates of the given points. A corresponding numerical method is developed and numerical examples are given.

## Zusammenfassung

Die HELMERT-Transformation wird in der Geodäsie gebraucht und angewandt. Eine spezielle affine Abbildung, bestehend aus einer Translation und einer mit einem Maßstabsfaktor multiplizierten Rotation, wird bestimmt, derart, dass eine gegebene Menge von Punkten im Raum näherungsweise auf eine zweite solche Menge abgebildet wird. Wir führen für jede der drei Punktkoordinaten einen eigenen Maßstabsfaktor ein. Ein auf diesem Fall zugeschnittenes numerisches Verfahren wird entwickelt. Numerische Beispiele werden angegeben.

## 1 Problem description

Let two sets  $(x_i, y_i, z_i)$  and  $(d_i, e_i, f_i)$ ,  $i = 1, \dots, m \geq 9$ , of spatial points be given. We look for a translation  $(a, b, c)$ , scale factors  $(u, v, w)$ , and angles  $(\alpha, \beta, \gamma)$  so that

$$\begin{pmatrix} d_i \\ e_i \\ f_i \end{pmatrix} \approx \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix} D(\alpha, \beta, \gamma) \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}, \quad (1)$$

$$i = 1, \dots, m.$$

Here

$$D(\alpha, \beta, \gamma) = D_1(\alpha)D_2(\beta)D_3(\gamma) \quad (2)$$

where

$$D_1(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (3)$$

$$D_2(\beta) = \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix}, \quad (4)$$

$$D_3(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & -\sin \gamma \\ 0 & \sin \gamma & \cos \gamma \end{pmatrix} \quad (5)$$

are elementary rotations in the  $x - y$ ,  $x - z$  and  $y - z$  plane, respectively. We assume that  $\det D(\alpha, \beta, \gamma) = +1$ ; the case of  $\det D(\alpha, \beta, \gamma) = -1$  can be treated by multiplying  $D(\alpha, \beta, \gamma)$  with  $\text{diag}(1, 1, -1)$  from the right. For  $u = v = w$  we have the classical HELMERT transformation used within geodesy. There are some well-known methods of treating this problem, see e.g. (Sanso 1973, Hansen and Norris 1981, Horn 1987, Horn et al. 1988), but those cannot be extended to different scale factors. If further  $u = v = w = 1$  is fixed you get back the problem treated in (Späth 2003). Different values for  $u, v$  and  $w$  were most likely not considered so far.

If the approximation  $\approx$  in (1) is done in the least squares sense, then we have to minimize

$$F(a, b, c, u, v, w, \alpha, \beta, \gamma) = \frac{1}{2} \sum_{i=1}^m [(d_i - u(\cos \alpha \cos \beta x_i - (\sin \alpha \cos \gamma + \cos \alpha \sin \beta \sin \gamma)y_i + (\sin \alpha \sin \gamma - \cos \alpha \sin \beta \cos \gamma)z_i) - a)^2 + (e_i - v(\sin \alpha \cos \beta x_i + (\cos \alpha \cos \gamma - \sin \alpha \sin \beta \sin \gamma)y_i - (\cos \alpha \sin \gamma + \sin \alpha \sin \beta \cos \gamma)z_i) - b)^2 + (f_i - w(\sin \beta x_i + \cos \beta \sin \gamma y_i + \cos \beta \cos \gamma z_i) - c)^2]. \quad (6)$$

Note that because of e.g.

$$\begin{aligned} F(a, b, c, u, v, w, \alpha, \beta, \gamma) &= F(a, b, c, u, v, w, \alpha - \pi, \pi - \beta, \gamma - \pi) \\ &= F(a, b, c, -u, -v, w, \alpha, \pi - \beta, \gamma - \pi) \end{aligned} \quad (7)$$

minima or global minima are not unique. Since  $F$  is continuous and bounded below their existence is not a problem.

Of course, one could introduce positive weights for each term in (6) or even different ones for each term in (6), i. e. for each coordinate. In order to keep the notation simple we will omit this.

## 2 A numerical method

Necessary conditions for a minimum of (6) are that the partial derivatives of  $F$  w. r. t. the nine unknowns  $a, b, c, u, v, w, \alpha, \beta, \gamma$  will vanish. Considering in turn

$$\frac{\partial F}{\partial a} = \frac{\partial F}{\partial u} = 0 \tag{8}$$

$$\frac{\partial F}{\partial b} = \frac{\partial F}{\partial v} = 0 \tag{9}$$

$$\frac{\partial F}{\partial c} = \frac{\partial F}{\partial w} = 0 \tag{10}$$

we get three systems of two linear equations with two variables  $(a, u)$ ,  $(b, v)$ , and  $(c, w)$ , respectively. The first system (8) reads

$$am + u \sum_{i=1}^m h_i = \sum_{i=1}^m d_i, \tag{11}$$

$$a \sum_{i=1}^m h_i + u \sum_{i=1}^m h_i^2 = \sum_{i=1}^m d_i h_i,$$

where

$$h_i = \cos \alpha \cos \beta x_i - (\sin \alpha \cos \gamma + \cos \alpha \sin \beta \sin \gamma) y_i + (\sin \alpha \sin \gamma - \cos \alpha \sin \beta \cos \gamma) z_i. \tag{12}$$

If at least two of the  $h_i$  are (most likely) different, then the solution  $(a, u)$  exists and is unique. Similar systems arise for (9) and (10). For (9) one has to replace in (11)  $(a, u)$  by  $(b, v)$ ,  $d_i$  by  $e_i$ , and  $h_i$  by

$$h_i = \sin \alpha \cos \beta x_i + (\cos \alpha \cos \gamma - \sin \alpha \sin \beta \sin \gamma) y_i - (\cos \alpha \sin \gamma + \sin \alpha \sin \beta \cos \gamma) z_i, \tag{13}$$

and for (10) one has to replace in (11)  $(a, u)$  by  $(c, w)$ ,  $d_i$  by  $f_i$ , and  $h_i$  by

$$h_i = \sin \beta x_i + \cos \beta \sin \gamma y_i + \cos \beta \cos \gamma z_i. \tag{14}$$

The remaining necessary conditions are

$$\frac{\partial F}{\partial \alpha} = \frac{\partial F}{\partial \beta} = \frac{\partial F}{\partial \gamma} = 0. \tag{15}$$

These three equations for the unknowns  $\alpha, \beta$ , and  $\gamma$  are highly nonlinear and cannot explicitly be solved as before. In this situation we do not try to solve (15) but will use direct minimization of  $F$  w.r.t.  $\alpha, \beta$ , and  $\gamma$ . This gives the following block descent method (Bertsekas 1995), that, in some different form, was also successfully used for the case  $u = v = w = 1$  (Späth 2003).

**Step 1:**

Choose starting values  $\alpha = \alpha^{(0)}, \beta = \beta^{(0)}, \gamma = \gamma^{(0)}$  with  $\alpha^{(0)}, \beta^{(0)}, \gamma^{(0)} \in [0, 2\pi)$ . Set  $t = 0$ .

**Step 2:**

For (11) – and the similar systems for (9) and (10) –, calculate the coefficients using (12), (13), (14) as described and solve the corresponding  $2 \times 2$  linear equations to get  $a^{(t)}, b^{(t)}, c^{(t)}, u^{(t)}, v^{(t)}, w^{(t)}$  as functions of  $\alpha^{(t)}, \beta^{(t)}, \gamma^{(t)}$ .

**Step 3:**  
Minimize

$$F_1(\alpha) = F(a^{(t)}, b^{(t)}, c^{(t)}, u^{(t)}, v^{(t)}, w^{(t)}, \alpha, \beta^{(t)}, \gamma^{(t)}) \tag{16}$$

by one-dimensional minimization w.r.t.  $\alpha$  to get  $\alpha^{(t+1)} = \alpha$ .

Minimize

$$F_2(\beta) = F(a^{(t)}, b^{(t)}, c^{(t)}, u^{(t)}, v^{(t)}, w^{(t)}, \alpha^{(t+1)}, \beta, \gamma^{(t)}) \tag{17}$$

to get  $\beta^{(t+1)} = \beta$ .

Minimize

$$F_3(\beta) = F(a^{(t)}, b^{(t)}, c^{(t)}, u^{(t)}, v^{(t)}, w^{(t)}, \alpha^{(t+1)}, \beta^{(t+1)}, \gamma) \tag{18}$$

to get  $\gamma^{(t+1)} = \gamma$ .

Set  $t := t + 1$  and go back to Step 2 if convergence has not occurred so far or stop if some maximal number of iterations has been exceeded.

The minimization of the functions  $F_1 = F_1(\alpha)$ ,  $F_2 = F_2(\beta)$ , and  $F_3 = F_3(\gamma)$  over  $[0, 2\pi)$  can easily be done using the subroutine FMIN (Forsythe, Malcolm, Moler 1977) with starting intervals  $[0, 2\pi)$  for  $\alpha, \beta$ , and  $\gamma$ , respectively, and a value  $TOL = 1.E - 5$  for the required accuracy of the result. In this way the implementation of the method requires about 160 lines of FORTRAN Code and additionally FMIN. Thus our method is easier to implement than the usual Gauss-Newton method that additionally needs the Jacobian.

The above algorithm is a descent method, but convergence to some minimum is not sure. It could be guaranteed if  $F$  would be convex and if  $F_1, F_2$ , and  $F_3$  would be strictly convex (Bertsekas 1995). Nevertheless, we empirically found out, even independently of the starting values for  $\alpha^{(0)}, \beta^{(0)}, \gamma^{(0)}$ , that the algorithm always found some (not unique) global optimum (see next section).

Note that similar as in (Späth 2003) the algorithm can be extended when using the  $L_1$ -norm instead of the squared  $L_2$ -norm in (6).

### 3 Numerical examples

To generate test data we used  $m = 16, a = 1, b = -3, c = 2, u = 2, v = 6, w = .5, \alpha = .5, \beta = 2$ , and  $\gamma = 4.5$  and further

$$\begin{array}{l|cccccccccccccccc} x_i & 1 & 0 & 2 & 0 & -1 & -3 & 4 & 2 & 5 & 0 & -2 & 3 & 0 & 5 & -4 & 1 \\ y_i & 0 & 2 & 2 & 0 & 1 & 3 & 3 & 8 & -1 & 0 & -2 & 0 & 0 & 6 & 2 & 1 \\ z_i & 2 & 3 & 0 & 1 & -4 & 4 & 1 & 3 & -1 & 5 & -3 & 0 & 4 & 7 & -6 & 5 \end{array}$$

to produce  $(d_i, e_i, f_i)$  using (1) and  $=$  instead of  $\approx$ . We allowed five digits after the fixed point in the decimal representation of  $(d_i, e_i, f_i)$ ,  $i = 1, \dots, m = 16$ . As starting values (among others) we always chose  $\alpha^{(0)} = 2.5$ ,  $\beta^{(0)} = 1$ ,  $\gamma^{(0)} = 5.5$ .

Using those data we got  $F = 0.000$  after 42 iterations and

$$\begin{aligned}(a, b, c) &= (1.000, -3.000, 2.000), \\(u, v, w) &= (-2.000, -6.000, .500), \\(\alpha, \beta, \gamma) &= (.500, 1.142, 1.356),\end{aligned}$$

i. e. a global minimum (see the remark at the end of Section 1).

For the next case we chopped all decimals except the first one after the fixed point. Here we got  $F = 0.034$  after 46 iterations and

$$\begin{aligned}(a, b, c) &= (.981, -3.001, 1.955), \\(u, v, w) &= (-1.987, -5.985, .501), \\(\alpha, \beta, \gamma) &= (.501, 1.143, 1.357),\end{aligned}$$

i. e. very plausible results for a global optimum, too.

Chopping the last remaining decimal after the fixed point, i. e. getting integers for  $(d_i, e_i, f_i)$ , we received  $F = 3.236$  after 53 iterations and

$$\begin{aligned}(a, b, c) &= (1.018, -3.072, 1.599), \\(u, v, w) &= (-1.836, -5.856, .481), \\(\alpha, \beta, \gamma) &= (.478, 1.139, 1.384).\end{aligned}$$

Finally, we alternatively added  $\pm 1$  to the integers received before. This resulted in

$$\begin{array}{l|cccccccccccccccc}d_i & 1 & 1 & 4 & 1 & 4 & 5 & 3 & 10 & -4 & -3 & -1 & 0 & -2 & 4 & 10 & 1 \\e_i & 6 & 17 & -3 & 1 & -24 & 26 & 1 & 24 & -15 & 26 & -19 & -7 & 20 & 38 & -28 & 24 \\f_i & 3 & 1 & 2 & 3 & 2 & 0 & 5 & 3 & 3 & 1 & -1 & 4 & 1 & 6 & -1 & 3\end{array}$$

Here we got after 137 iterations  $F = 22.786$  and

$$\begin{aligned}(a, b, c) &= (.745, -3.103, 1.351), \\(u, v, w) &= (-1.727, -5.847, .584), \\(\alpha, \beta, \gamma) &= (.625, 1.211, 1.231).\end{aligned}$$

Because of continuity we think we have got a global minimum in every case.

## 4 Conclusions

Extending the determination of the well-known spatial HELMERT transformation to the case of different scale factors for each of the three coordinates requires a new method. An iterative algorithm was developed that works very well for test data of different accuracy. Nevertheless, convergence could not be proved so far.

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