The BLUE of the GPS Double Difference Satellite-to-Receiver Range for Precise Positioning

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Summary

The standard technique for GPS network adjustment is to use the L1 phase observable for short baselines and the ionosphere-free linear combination of L1 and L2 phase observables for long baselines. The definition of short and long baseline is subjective, and the resulting observable may be quite different for intermediate baselines dependent on the choice.

We derive and propose the use of the Best Linear Unbiased Estimator (BLUE) of the combination of phase and code observations as well as an a priori ionosphere bias. The BLUE is always at least as accurate as and usually better than any of the previous observations, and it does not necessitate the artificial division of the adjustment scheme into short and long baselines.

Zusammenfassung

Die Standardtechnik zur Ausgleichung von GPS-Netzen besteht bei kurzen Basislinien in der Nutzung der L1-Phasenmessung und bei langen Basislinien in der Verarbeitung der ionosphärischen Linearkombination der L1- und L2-Phasenbeobachtungen. Die Definition einer kurzen und einer langen Basislinie ist jedoch subjektiv, und das Ergebnis kann sich bei Basislinien mittlerer Länge in Abhängigkeit vom Ansatz erheblich unterscheiden.

Wir leiten hier einen besten linearen erwartungstreuen Schätzwert (Best Linear Unbiased Estimator; BLUE) für die Kombination von Phasen- und Code-Messungen sowie einen a priori Ionosphärenparameter ab. Dieser BLUE ist mindestens so genau und im Allgemeinen besser als eine der vorangegangenen Observablen, die künstliche Unterscheidung zwischen der Ausgleichung kurzer und langer Basislinien entfällt.

1 Introduction

Today dual frequency GPS is essential for precise position, particularly for real time kinematic (RTK) applications. In all such applications the fixing of the carrier phase ambiguities for both frequencies is crucial. The success of this initialisation step is closely related with the significance of the ionosphere bias. Using the double difference observable, the magnitude of the ionosphere bias is reduced and dependent on the baseline length. As a consequence, there is always a critical baseline length, say 15 km, from which the ionosphere bias starts to affect the estimates of position. From dual frequency code and phase GPS data the Best Linear Unbiased Combination (BLUE) of phase ambiguities as well as satellite-to-receiver range can be formulated. Sjöberg (1999a,b) discussed the problems stemming from the ionosphere bias with the $BLUE_1$ (assuming no ionosphere bias) and $BLUE_2$ (including modelling of the ionosphere bias). If there is some a priori information on the ionosphere bias, a biased estimator, the Restricted Best Linear Estimator (RBLE), may be a better compromise between random noise and ionosphere bias than any of these unbiased estimators.

The studies of Sjöberg (1999a,b) were restricted to the fixing of phase ambiguities. This study will extend the investigation to the estimation of the satellite-to-receiver range , which is the most essential quantity for the determination of the coordinates (x, y, z) of a wanted position by the basic formula

$$\rho = \sqrt{(x-x^s)^2+(y-y^s)^2+(z-z^s)^2}\ , \tag{1}$$

where (x^s, y^s, z^s) is the position of the satellite. In addition, we will throughout assume that the a priori information of the ionosphere bias is available as an observation (which might be zero) along with its standard error. As we shall see, this has the advantage that we can formulate improved unbiased estimators for ambiguities and range.

2 Observation equations and general estimators

From the data recorded by a pair of dual frequency phase and code observing GPS receivers, the following observation equations can be formulated:

$$\begin{aligned} l_1 &= \varphi_1\lambda_1 = \rho + \lambda_1 N_1 - \frac{I}{f_1^2} + \varepsilon_{11} \\ l_2 &= \varphi_2\lambda_2 = \rho + \lambda_2 N_2 - \frac{I}{f_2^2} + \varepsilon_{12} \\ l_3 &= R_1 = \rho + \frac{I}{f_1^2} + \varepsilon_{21} \\ l_4 &= R_2 = \rho + \frac{I}{f_2^2} + \varepsilon_{22}, \end{aligned} \tag{2}$$

where φ_i and λ_i are the carrier phase observations and wavelengths, respectively, f_i and N_i the corresponding frequencies and integer phase ambiguities, I the ionosphere bias and, finally, ε_{ij} are random, uncorrelated observation errors. We will also assume that the data are double differences (i. e. including two GPS receivers and two satellites) and that the standard errors of the phase observables (l_1 and l_2) are 6 mm and of code observables

(l_3 and l_4) 60 cm. Hence, for later use we introduce the ratio

$$k = (\sigma_{\varphi\lambda} / \sigma_R)^2 = 1x10^{-4} \ . \tag{3}$$

Our primary interest is to determine the satellite-to-receiver range ρ in real-time, i. e. from only one epoch of data. The system of four equations (2) contains exactly four unknowns, and the direct solution for ρ becomes

$$\hat{\rho} = \frac{\nu^2 R_1 - R_2}{\nu^2 - 1} \ , \tag{4a}$$

where

$$\nu = (f_1 / f_2)^2 = (\lambda_2 / \lambda_1)^2 \tag{4b}$$

with λ_1 = 19.0 cm and λ_2 = 24.4 cm, yielding the following standard error of the estimator (4a):

$$\sigma_{\hat{\rho}} = \frac{\sqrt{\nu^4 + 1}}{\nu^2 - 1} \sigma_R \ , \tag{5}$$

with the estimated value 1.78 m for previous value of σ_R. This estimate for ρ is too poor for precise positioning.

One way to improve the estimate is to augment the system (2) by some a priori unbiased estimate of the ionosphere bias:

$$l_5 = \frac{I}{f_1^2} + \varepsilon_I \ , \tag{6}$$

where ε_I is a random error. For example, it is well known that the ionosphere bias for double difference observable decreases with baseline length, so that for baselines within, say, 15 km l_5 can be set to zero with a certain standard error.

All the information contained in Eqs. (2) and (6) can be merged into the following general estimator of ρ:

$$\tilde{\rho} = \sum_{i=1}^{5} a_i l_i \ , \tag{7}$$

where a_i are arbitrary constants. Inserting Eqs. (2) and (6) one can rewrite the estimator as

$$\begin{aligned} \tilde{\rho} = \rho \sum_{i=1}^{4} a_i + a_1 \lambda_1 N_1 + a_2 \lambda_2 N_2 + \\ + \frac{I}{f_1^2}(-a_1 - a_2 \nu^2 + a_3 + a_4 \nu^2 + a_5) + d\tilde{\rho} \ , \end{aligned} \tag{8a}$$

where

$$d\tilde{\rho} = a_1 \varepsilon_{11} + a_2 \varepsilon_{12} + a_3 \varepsilon_{21} + a_4 \varepsilon_{22} + a_5 \varepsilon_I \tag{8b}$$

is the random error of $\tilde{\rho}$. It follows directly from Eq. (8a) that it provides an unbiased estimator of ρ if and only if the following conditions are satisfied:

$$\sum_{i=1}^{4} a_i = 1 \ , \tag{9a}$$

$$a_1 N_1 \lambda_1 + a_2 \lambda_2 N_2 = 0 \tag{9b}$$

and

$$-a_1 - a_2 \nu^2 + a_3 + a_4 \nu^2 + a_5 = 0 \ . \tag{9c}$$

In order to fulfil Eq. (9b) one either has to set a_1 and a_2 to zero (implying that the high quality phase data are rejected), or one or both of the integer ambiguities must be known. Hence, a first goal must be to determine these ambiguities before estimating ρ. This is not always possible in real-time applications, and such a failure will inevitably lead to a less accurate estimate of ρ. Below we will study the Best Linear Unbiased Estimators (BLUEs) of ρ under various conditions of knowing the ambiguities. All BLUEs require that Eq. (9c) be satisfied, implying that we are only interested in so-called »ionosphere-free« linear combinations.

3 Short baselines

Short baselines we define as baselines so short that the ionosphere bias can be omitted in the basic observation model (2). If this is the case, the estimation of ρ improves significantly. Then one can also fix the cycle ambiguities within merely one epoch of observations (Sjöberg 1996, 1998 and Horemuz and Sjöberg 2002), and ρ becomes the only remaining unknown of Eq. (2). In this case, ρ is frequently solved only from the L_1 phase observable, yielding the estimator

$$\hat{\rho}_1 = l_1 - \lambda_1 N_1 \tag{10}$$

with the standard error $\sigma_{\varphi\lambda}$, which is 6 mm according to previous assumption.

However, this is not the best choice, but useful information is omitted. A better choice is the BLUE which is achieved by minimizing the variance of the general estimator (7). However, if N_1 and N_2 are known, the observables l_3 and l_4 will not add any significant information due to there relatively large standard errors, and the BLUE is given by

$$\hat{\rho}_{BLUE} = \frac{l_1 - \lambda_1 N_1 + l_2 - \lambda_2 N_2}{2} \tag{11}$$

with the standard error $\sigma_{\varphi\lambda} / \sqrt{2}$, i. e. only 71 % of the error of $\hat{\rho}_1$. The above estimators guarantee a quick and precise positioning suitable for real-time applications.

For comparison we now study also the case when integer ambiguities have not been fixed. In this case, the general estimator (7) includes the first four observables only. Eq. (8a) does not include the ionosphere bias term and the coefficients a_1 and a_2 must be set to zero (i.e. no use of phase observables) in order to eliminate the ambiguity bias. As a result, the BLUE becomes

$$\hat{\rho}_{34} = (R_1 + R_2)/2 \tag{12}$$

with standard error $\sigma_R/\sqrt{2}$, which is 42 cm or 100 times bigger than for the estimator of Eq. (11). This shows once again the dramatic improvement gained by first solving the ambiguities.

4 Long baselines with N_w fixed

For long baselines, the effect of the ionosphere bias hampers the solution to phase ambiguities as well as satellite-to-receiver range. Nevertheless, the wide-lane ambiguity $N_w = N_1 - N_2$ is easy to resolve also in this case from the dual frequency code and phase GPS data (Wübbena 1985; Sjöberg 1998 and 1999a,b; Horemuz and Sjöberg 2002). Hence, we can assume that N_w is known, which can be used to eliminate N_2 from the system of equations (2) plus (6). The resulting system can be written in matrix notations

$$\mathbf{AX} = \mathbf{L} - \varepsilon , \tag{13a}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & \nu & -\nu^2 \\ 1 & & 1 \\ 1 & \nu^2 & \\ & & 1 \end{bmatrix}; \ \mathbf{X} = \begin{bmatrix} \rho \\ \lambda_1 N_1 \\ I/f_1^2 \end{bmatrix} \text{ and } \mathbf{L} = \begin{bmatrix} l_1 \\ l_2 - \lambda_2 N_2 \\ l_3 \\ l_4 \\ l_5 \end{bmatrix} \tag{13b}$$

and ε is the vector of random errors of $\mathbf{L}$. The system (13) has five equations and three unknowns. Its least squares solution (or BLUE) for $\mathbf{X}$ (to order k) reads

$$\hat{\mathbf{X}} = \mathbf{N}^{-1}\mathbf{A}^T\mathbf{Q}^{-1}\mathbf{L} = \mathbf{N}^{-1}\mathbf{h} \tag{14a}$$

and

$$\mathbf{Q}_{\hat{x}\hat{x}} = \frac{\sigma_{\varphi\lambda}^2}{\det} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}, \tag{14b}$$

where

$$\det = p(\nu-1)^2 + k[2p + (\nu-1)^4](\nu^2+1) \tag{14c}$$

$$\begin{aligned} C_{11} &= \nu^2(\nu-1)^2 + p(\nu^2+1) + k(\nu^4+1)(\nu^2+1) \\ C_{12} &= (\nu-1)^3(\nu+1) - p(\nu+1) - \\ &\quad -k(2+\nu+\nu^2+\nu^3+\nu^4+2\nu^5) \\ C_{13} &= -\nu(\nu-1)^2 - k(\nu^2+1)^2 \\ C_{22} &= (\nu^2-1)^2 + 2p + 2k(3+p+2\nu^2+3\nu^3) \\ C_{23} &= (\nu-1)^2(\nu+1) + k(3+\nu+\nu^2+3\nu^3) \\ C_{33} &= (\nu-1)^2 + 2k(\nu^2+1) \end{aligned} \tag{14d}$$

$$\begin{aligned} h_1 &= l_1 + l_2 + k(l_3 + l_4) \\ h_2 &= l_1 + \nu l_2 \\ h_3 &= -l_1 - \nu^2 l_2 + k l_3 + k\nu^2 l_4 + p l_5 \end{aligned} \tag{14e}$$

and

$$k = (\sigma_{\varphi\lambda}/\sigma_R)^2$$
$$p = (\sigma_{\varphi\lambda}/\sigma_I)^2 \ .$$

In particular, we obtain the standard errors

$$\sigma_{\hat{\rho}} = \sigma_{\varphi\lambda}\sqrt{C_{11}/\det} \tag{15a}$$

$$\sigma_{\hat{N}_1} = \frac{\sigma_{\varphi\lambda}}{\lambda_1}\sqrt{\frac{C_{22}}{\det}} \tag{15b}$$

and

$$\sigma_{I/f_1^2} = \sigma_{\varphi\lambda}\sqrt{C_{33}/\det} \ , \tag{15c}$$

which, in the limit $\sigma_I \to 0$ (and $p \to \infty$; no ionosphere bias), yield (to order k)

$$\sigma_{\hat{\rho}} \to \sigma_{\varphi\lambda}\frac{\sqrt{\nu^2+1}}{\nu-1} \tag{16a}$$

$$\sigma_{\hat{N}_1} \to \frac{\sigma_{\varphi\lambda}}{\lambda_1}\frac{\sqrt{2}}{\nu-1} \tag{16b}$$

and

$$\sigma_{\hat{I}/f_1^2} \to 0 \ . \tag{16c}$$

If $\sigma_{\varphi\lambda}$ is set to 6 mm this implies that the limiting values for the standard errors of $\hat{\rho}$ and $\hat{N}_1$ become 34 mm and 0.16, respectively. Hence, in this limit all unknowns are well determined already after a single epoch data.

The other extreme is the case when σ_I approaches infinity (i.e. no a priori information about the ionosphere bias). Then one obtains the standard errors

$$\sigma_{\hat{\rho}} = \sigma_R \frac{\nu}{(\nu-1)\sqrt{\nu^2+1}} \tag{17a}$$

$$\sigma_{\hat{N}_1} = \frac{\sigma_R(\nu+1)}{\lambda_1(\nu-1)\sqrt{\nu^2+1}} \quad (17b)$$

$$\sigma_{I/f_1^2} = \frac{\sigma_R}{(\nu-1)\sqrt{\nu^2+1}} \; . \quad (17c)$$

For σ_R set to 60 cm this yields the standard errors 1.67 m, 15.6 and 1.30 m for Eqs. (17a)–(17c), which tells us that for no a priori information about the ionosphere bias, the wide-lane ambiguity does not help much in estimating ρ and N_1 compared to the original estimators from model (2).

We may thus conclude that the a priori information about the ionosphere bias is very critical for the outcome of the estimators. The ratio between the posterior and a priori variance of the ionosphere bias can be written

$$\left(\frac{\sigma_{\hat{I}/f_1^2}}{\sigma_I}\right)^2 = \frac{p[(\nu-1)^2+2k(\nu^2+1)]}{p(\nu-1)^2+k[2p+(\nu-1)^4](\nu^2+1)} \; , \quad (18)$$

which is limited by

$$0 \le \left(\frac{\sigma_{\hat{I}/f_1^2}}{\sigma_I}\right)^2 \le 1 \; . \quad (19)$$

The lower limit is attained for a complete prior knowledge of the ionosphere bias, and the upper limit reflects no prior information about the bias. Consequently, significant prior information about the ionosphere bias can be very advantageous for determining the unknowns of Eq. (13a). If we require that $\sigma_{\hat{N}_1}$ be within 0.3 to be able to fix N_1 to the correct integer, it can be shown from Eq. (15b) with $\sigma_{\varphi\lambda} = 6$ mm, σ_I must be within 6 cm. This lower limit for σ_I is actually 1.9 times larger than previous limit presented by Blewitt (1989) and Sjöberg (1999a) using a combination of wide-lane and ionosphere-free linear combination to estimate the ambiguity of L_1.

5 Long baselines with N_1 and N_2 fixed

If N_1 and N_2 are fixed, it implies that the ambiguities can be removed from the unknowns of Eq. (2), and the quality of the estimate of ρ improves considerably also for long baselines. Due to the high quality of the phase observables compared to code observables, the latter will not contribute much to the solution for ρ (or I), but the BLUE is well estimated by

$$\hat{\rho} = \frac{\nu^2\lambda_1(\phi_1-N_1)-\lambda_2(\varphi_2-N_2)}{\nu^2-1} \quad (20)$$

with the standard error

$$\sigma_{\hat{\rho}} = \sigma_{\varphi\lambda}\frac{\sqrt{\nu^4+1}}{\nu^2-1} \; , \quad (21)$$

which, for a standard error of 6 mm for the phase observation, becomes 18 mm.

We now compare the estimates (11) and (20). Both are unbiased linear combinations of L_1 and L_2 phase observations. However, Eq. (11), which has a more than four times smaller standard error, assumes that the ionosphere bias (I) is negligible, while estimator (20) includes the modelling of I. Hence, the outcome of whether we model the ionosphere or not is quite significant. Typically, GPS software manufacturers provide a default choice $I = 0$ for baselines within 15–20 km, while I is included as an unknown for longer baselines.

We now consider the case when there is additional information (l_5) on the ionosphere bias. Then the system of equations can be written

$$\mathbf{AX} = \mathbf{L} - \varepsilon \; , \quad (22a)$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & -\upsilon^2 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} \rho \\ I/f_1^2 \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} l_1 \\ l_2 \\ l_5 \end{bmatrix} \quad (22b)$$

$$l_i = (\phi_i - N_i)\lambda_i \quad \text{for} \quad i = 1,2 \quad (22c)$$

and

$$E\{\varepsilon\varepsilon^T\} = \mathbf{Q} = \sigma^2{}_{\varphi\lambda} \bullet \text{diagonal}(1,1,p^{-1}) \quad (22d)$$

with

$$p = (\sigma_{\phi\lambda}/\sigma_I)^2 \; . \quad (22e)$$

The least squares solution to this system of equations becomes

$$\hat{X} = N^{-1}A^TQ^{-1}L \quad (23a)$$

with the covariance matrix

$$Q_{\hat{X}\hat{X}} = \sigma^2_{\varphi\lambda}N^{-1} \; , \quad (23b)$$

where

$$N^{-1} = (A^TQ^{-1}A)^{-1} = \det{}^{-1}\begin{bmatrix} 1+\nu^4+p & 1+\nu^2 \\ 1+\nu^2 & 2 \end{bmatrix} \quad (23c)$$

and

$$\det = (\nu^2-1)^2 + 2p \; . \quad (23d)$$

The explicit solution for ρ can be expressed by

$$\hat{\rho} = a_1 l_1 + a_2 l_2 + a_5 l_5 \ , \tag{24a}$$

where

$$\begin{aligned} a_1 &= (\nu^4 - \nu^2 + p)/\det \\ a_2 &= (1 - \nu^2 + p)/\det \\ a_5 &= p(1 + \nu^2)/\det \end{aligned} \tag{24b}$$

with the standard error

$$\sigma_{\hat{\rho}} = \sigma_{\varphi\lambda} \sqrt{\frac{1 + \nu^4 + p}{(\nu^2 - 1)^2 + 2p}} \ . \tag{25}$$

This estimator for ρ satisfies the conditions (9a)–(9c) for unbiased estimation, and, as it also minimizes the expected variance, it is a BLUE.

The following special cases of the standard error (25) are of particular interest.

Case 1 (No ionosphere effect): As σ_I approaches zero, p goes to infinity and it follows that

$$\sigma_{\hat{\rho}} \to \sigma_{\varphi\lambda} / \sqrt{2} \quad \text{as} \quad p \to \infty \ , \tag{26}$$

which is the same as for the BLUE (11) for no ionosphere effect.

Case 2 (The ionosphere bias is completely unknown):

$$\sigma_{\hat{\rho}} \to \sigma_{\varphi\lambda} \frac{\sqrt{1 + \nu^4}}{\nu^2 - 1} \quad \text{as} \quad p \to 0 \ . \tag{27}$$

which agrees with the standard error (21) of the so-called »ionosphere-free linear combination«. The standard error of the BLUE (24a,b) is thus limited by the following inequalities

$$\sigma_{1,2} = \frac{\sigma_{\varphi\lambda}}{\sqrt{2}} \le \sigma_{\hat{\rho}} \le \sigma_{\varphi\lambda} \frac{\sqrt{1 + \nu^4}}{\nu^2 - 1} = \sigma_{\hat{\rho}_3} \ , \tag{28}$$

where the lower limit prevails for no ionosphere bias (or completely known bias) and the upper limit holds for a significant but unknown ionosphere bias. Hence, $\hat{\rho}$ is the BLUE that yields a smooth transition between solutions for various degrees of knowledge of the ionosphere bias.

6 Conclusions

Standard software for GPS network adjustment discriminates between short and long baselines. For short baselines the adjustment is typically carried out by the L1 phase observation alone, while for long baselines the ionosphere-free linear combination L3 of the L1 and L2 phase observations is utilized. This implies that sub-optimal estimators are used for most baselines. For short baselines high quality L2 phase data is discarded, and for long baselines the L3 solution is usually less accurate than the BLUE, including some a priori information of the ionosphere bias. In addition, the BLUE of the satellite-to-receiver range, derived in this article, can be directly used as the homogeneous observable for all baselines. Hence, the less successful adjustment strategy of dividing the network into short and long baselines with different types of observables should be avoided.

References

Blewitt, G.: Carrier phase ambiguity resolution for GPS applied to geodetic baselines up to 2000 m. J. of Geophysics, 94, B8: 10187–10203, 1989.

Horemuz, M. and L. E. Sjöberg: Rapid GPS ambiguity resolution for short and long baselines. J. Geod., 76: 381-391, 2002.

Sjöberg, L. E.: Application of GPS in detailed surveying, ZfV, 121: 485–491, 1996.

Sjöberg, L. E.: A new method for GPS ambiguity resolution by combined phase and code observables, Survey Review, 34, 268: 363–372, 1998.

Sjöberg, L. E.: The ionospheric problem in GPS ambiguity resolution and some possible solutions. In Bo Jonsson (Ed.): Proc 13th general meeting of the Nordic Geodetic Commission, Gävle, May 1998, part 2, pp. 320–330, Reports in Geodesy and Geographic Information Systems, The National land Survey of Sweden, Gävle, 1999a.

Sjöberg, L. E.: Unbiased vs. biased estimation of GPS phase ambiguities from dual-frequency code and phase observables, J. Geod., 73: 118–124, 1999b.

Wübbena, G (1985): Software developments for geodetic positioning with GPS using TI4100 code and carrier measurements. In C. Gode (Ed.): Proc. 1st Int. Symp. on Precise Positioning with the GPS, U. S. Dept. of Commerce, Rockwille, Md, pp. 403–412, 1985.

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